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Conformal spectra, moduli spaces and the Friedlander-Nadirashvili invariants

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Conformal spectra, moduli spaces and the Friedlander-Nadirashvili invariants

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Résumé

Dans cette thèse, nous étudions le spectre conforme d'une surface fermée et le spectre de Steklov conforme d'une surface compacte à bord et leur application à la géométrie conforme et à la topologie. Soit (Σ, c) une surface fermée munie d'une classe conforme c. Alors la $k-$ième valeur propre conforme est définie comme

$$\Lambda_k(\Sigma,c) = \sup_{g \in c}(\lambda_k(g) \cdot Aire(\Sigma,g)),$$

où $\lambda_k(g)$ est la $k-$ième valeur propre de l'operateur de Laplace-Beltrami de la métrique g sur Σ. Notons que nous commeçons par $\lambda_0(g) = 0$. En prennant le supremum sur toutes les classes conformes $\mathcal{C}$ sur Σ on obtient l'invariant topologique suivant de Σ:

$$\Lambda_k(\Sigma) = \sup_{c \in \mathcal{C}} \Lambda_k(\Sigma,c).$$

D'après l'article [**65**], les quantités $\Lambda_k(\Sigma, c)$ et $\Lambda_k(\Sigma)$ sont bien définies. Si une métrique g sur Σ satisfait

$$\lambda_k(g) \cdot Aire(\Sigma, g) = \Lambda_k(\Sigma),$$

alors on dit que g est maximale pour la fonctionnelle $\lambda_k(g) \cdot Aire(\Sigma, g)$. Dans l'article [**73**], il a été montré que les métriques maximales pour $\lambda_1(g) \cdot Aire(\Sigma, g)$ peuvent au pire avoir des singularités coniques. Dans cette thèse nous montrons que les métriques maximales pour les fonctionnelles $\lambda_1(g) \cdot Aire(\mathbb{T}^2, g)$ et $\lambda_1(g) \cdot Aire(\mathbb{KL}, g)$, où $\mathbb{T}^2$ et $\mathbb{KL}$ dénotent le $2-$tore et la bouteille de Klein, ne peuvent pas avoir de singularités coniques. Ce résultat découle d'un théorème de classification de classes conformes par des métriques induites d'une immersion minimale ramifiée dans une sphère ronde aussi montré dans cette thése.

Un autre invariant que nous étudions dans cette thèse est le $k-$ième invariant de Friedlander-Nadirashvili défini comme:

$$I_k(\Sigma) = \inf_{c \in \mathcal{C}} \Lambda_k(\Sigma, c)$$

L'invariant $I_1(\Sigma)$ a été introduit dans l'article [**34**]. Dans cette thèse nous montrons que pour toute surface orientable et pour toute surface non-orientable de genre impaire $I_k(\Sigma) = I_k(\mathbb{S}^2)$ et pour toute surface non-orientable de genre paire $I_k(\mathbb{RP}^2) \geq I_k(\Sigma) > I_k(\mathbb{S}^2)$. Ici $\mathbb{S}^2$ et $\mathbb{RP}^2$

dénotent la $2-$sphère et le plan projectif. Nous conjecturons que $I_k(\Sigma)$ sont des invariants des cobordismes des surfaces fermées.

Le spectre de Steklov conforme est défini de manière similaire. Soit (Σ, c) une surface compacte à bord non vide $\partial\Sigma$, alors les $k-$ièmes valeurs propres de Steklov conformes sont définies comme:

$$\sigma_k^*(\Sigma, c) = \sup_{g \in c}(\sigma_k(g) \cdot Longueur(\partial\Sigma, g)),$$

où $\sigma_k(g)$ est la $k-$ième valeur propre de Steklov de la métrique g sur Σ. Ici nous supposons que $\sigma_0(g) = 0$.

De façon similaire au problème fermé, on peut définir les quantités suivantes:

$$\sigma_k^*(\Sigma) = \sup_{c \in \mathcal{C}} \sigma_k^*(\Sigma, c).$$

et

$$I_k^\sigma(\Sigma) = \inf_{c \in \mathcal{C}} \sigma_k^*(\Sigma, c).$$

Les résultats de l'article [**16**] impliquent que toutes ces quantités sont bien définies. Dans cette thèse on obtient une formule pour la limite de $\sigma_k^*(\Sigma, c_n)$ lorsque la suite des classes conformes c_n dégénère. Cette formule implique que pour toute surface á bord $I_k^\sigma(\Sigma) = I_k^\sigma(\mathbb{D}^2)$, où $\mathbb{D}^2$ dénote le $2-$disque. On remarque aussi que les quantités $I_k^\sigma(\Sigma)$ sont des invariants des cobordismes de surfaces à bord. De plus, on obtient une borne supérieure pour la fonctionnelle $\sigma_k(g) \cdot Longueur(\partial\Sigma, g)$, où Σ est non-orientable, en terme de son genre et le nombre de composants de bord.

Mots-clés : géométrie spectrale, immersions minimales ramifiées, métriques à singularités coniques, métriques maximales, spectre conforme, invariants de Friedlander-Nadirashvili, espace des modules, cobordismes, spectre de Steklov conforme, bornes supérieures.

Abstract

In this thesis, we study the conformal spectrum of a closed surface and the conformal Steklov spectrum of a compact surface with boundary and their application to conformal geometry and topology. Let (Σ,c) be a closed surface endowed with a conformal class c then the $k-$th conformal eigenvalue is defined as

$$\Lambda_k(\Sigma,c) = \sup_{g\in c}(\lambda_k(g)\cdot Area(\Sigma,g)),$$

where $\lambda_k(g)$ is the $k-$th Laplace-Beltrami eigenvalue of the metric g on Σ. Note that we start with $\lambda_0(g) = 0$. Taking the supremum over all conformal classes $\mathcal{C}$ on Σ one gets the following topological invariant of Σ:

$$\Lambda_k(\Sigma) = \sup_{c\in\mathcal{C}} \Lambda_k(\Sigma,c).$$

It follows from the paper [**65**] that the quantities $\Lambda_k(\Sigma,c)$ and $\Lambda_k(\Sigma)$ are well-defined. Suppose that for a metric g on Σ the following identity holds

$$\lambda_k(g)\cdot Area(\Sigma,g) = \Lambda_k(\Sigma).$$

Then one says that the metric g is maximal for the functional $\lambda_k(g)\cdot Area(\Sigma,g)$. In the paper [**73**] it was shown that the maximal metrics for the functional $\lambda_1(g)\cdot Area(\Sigma,g)$ at worst can have conical singularities. In this thesis we show that the maximal metrics for the functionals $\lambda_1(g)\cdot Area(\mathbb{T}^2,g)$ and $\lambda_1(g)\cdot Area(\mathbb{KL},g)$, where $\mathbb{T}^2$ and $\mathbb{KL}$ stand for the $2-$torus and the Klein bottle respectively, cannot have conical singularities. This result is a corollary of a conformal class classification theorem by metrics induced from a branched minimal immersion into a round sphere that we also prove in the thesis.

Another invariant that we study in this thesis is the $k-$th Friedlander-Nadirashvili invariant defined as:

$$I_k(\Sigma) = \inf_{c\in\mathcal{C}} \Lambda_k(\Sigma,c).$$

The invariant $I_1(\Sigma)$ was introduced in the paper [**34**]. In this thesis we prove that for any orientable surface and any non-orientable surface of odd genus $I_k(\Sigma) = I_k(\mathbb{S}^2)$ and for any non-orientable surface of even genus $I_k(\mathbb{RP}^2) \geq I_k(\Sigma) > I_k(\mathbb{S}^2)$. Here $\mathbb{S}^2$ and $\mathbb{RP}^2$ denote the

2−sphere and the projective plane respectively. We also conjecture that $I_k(\Sigma)$ are invariants of cobordisms of closed manifolds.

The conformal Steklov spectrum is defined in a similar way. Let (Σ, c) be a compact surface with non-empty boundary $\partial\Sigma$ then the $k-$th conformal Steklov eigenvalues is defined by the formula:

$$\sigma_k^*(\Sigma,c) = \sup_{g\in c}(\sigma_k(g) \cdot Length(\partial\Sigma,g)),$$

where $\sigma_k(g)$ is the $k-$th Steklov eigenvalue of the metric g on Σ. Here we suppose that $\sigma_0(g) = 0$.

Similarly to the closed problem one can define the following quantities:

$$\sigma_k^*(\Sigma) = \sup_{c\in\mathcal{C}} \sigma_k^*(\Sigma,c)$$

and

$$I_k^\sigma(\Sigma) = \inf_{c\in\mathcal{C}} \sigma_k^*(\Sigma,c).$$

The results of the paper [**16**] imply that all these quantities are well-defined. In this thesis we obtain a formula for the limit of the $k-$th conformal Steklov eigenvalue when the sequence of conformal classes degenerates. Using this formula we show that for any surface with boundary $I_k^\sigma(\Sigma) = I_k^\sigma(\mathbb{D}^2)$, where $\mathbb{D}^2$ stands for the 2−disc. We also notice that $I_k^\sigma(\Sigma)$ are invariants of cobordisms of surfaces with boundary. Moreover, we obtain an upper bound for the functional $\sigma_k(g) \cdot Length(\partial\Sigma,g)$, where Σ is non-orientable, in terms of its genus and the number of boundary components.

Keywords : Spectral geometry, branched minimal immersions, metrics with conical singularities, maximal metrics, conformal spectrum, the Friedlander-Nadirashvili invariants, moduli space, cobordisms, conformal Steklov spectrum, upper bounds.

Contents

List of figures

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Introduction

Spectral geometry is a branch of *geometric analysis,* a modern area of mathematics, which studies the interaction of analysis, geometry and topology of a Riemannian manifold. The main research object of spectral geometry is the spectrum of some globally defined (by the Riemannian metric) operator on a Riemannian manifold. The spectrum of such an operator is a natural invariant of the Riemannian manifold as well as the curvature, geodesics and minimal submanifolds. Moreover, it turns out that the spectrum contains a lot of geometric information about the manifold.

The *Laplace-Beltrami operator*, which is often called *the Laplacian*, is the most important operator in Riemannian geometry. It is defined in the following way. Let (M,g) be a closed Riemannian manifold. Then the Laplacian, denoted by Δ_g, is defined as $-div \circ grad$. In the local coordinates x_i this operator can be expressed by the formula

$$\Delta_g = -\frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x_i}\Big(g^{ij}\sqrt{\det(g)}\frac{\partial}{\partial x_i}\Big).$$

It is a self-adjoint elliptic differential operator of second order. Its spectrum is a discrete collection of non-negative eigenvalues of finite multiplicities with the only accumulation point at infinity:

$$0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g) \leqslant \ldots \nearrow +\infty.$$

If the Riemannian manifold (M,g) is compact with non-empty boundary then one can consider the following eigenvalue problems:

- The *Dirichlet problem*:

$$\begin{cases} \Delta_g u = \lambda^D u & \text{in } M, \\ u = 0 & \text{on } \partial M. \end{cases} \tag{0.0.1}$$

  The set of all λ^D for which this problem admits a solution is often called the *Dirichlet spectrum.* It is a discrete collection of positive numbers of finite multiplicities with the accumulation point at infinity:

$$0 < \lambda_1^D(g) \leqslant \lambda_2^D(g) \leqslant \ldots \nearrow +\infty.$$

- The *Neumann problem*:

$$\begin{cases} \Delta_g u = \lambda^N u & \text{in } M, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial M. \end{cases} \tag{0.0.2}$$

  Similarly, the set of all λ^N such that problem (0.0.2) admits a solution is called the *Neumann spectrum*. It is a discrete collection of non-negative numbers of finite multiplicities with the accumulation point at infinity:

$$0 = \lambda_0^N(g) < \lambda_1^N(g) \leqslant \lambda_2^N(g) \leqslant \ldots \nearrow +\infty.$$

- The *Steklov problem*:

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial M. \end{cases} \tag{0.0.3}$$

  We refer to the set of all σ for which the Steklov problem admits a solution as to the *Steklov spectrum*. It is a discrete collection of non-negative numbers of finite multiplicities with the accumulation point at infinity:

$$0 = \sigma_0(g) < \sigma_1(g) \leqslant \sigma_2(g) \leqslant \ldots \nearrow +\infty.$$

Thus, all these spectra are invariants of a given Riemannian manifold (M,g). However, the manifold M carries infinitely many different metrics. Then one can ask the following natural question: how big and how small can the $k-$th eigenvalue of one of the aforesaid problems on a given manifold M be if we allow the metric to change? The branch of spectral geometry which tries to answer this question is called the *geometric optimization of eigenvalues.*

0.0.1. Geometric optimization of eigenvalues

Apparently the first question in geometric optimization of eigenvalues was asked by baron Rayleigh [**101**]. In his famous book "The Theory of Sound" he asks: find a planar domain which minimizes the first eigenvalue of the Dirichlet problem. Without fixing the area of the domain this problem becomes trivial since under the dilatations by t the spectrum is divided by t^2 and hence λ_1^D can be made arbitrarily small. If we fix the area of the domain then the problem becomes scale invariant. Rayleigh gave the correct answer to his question and provided some evidences: the minimum of the first Dirichlet eigenvalue among all planar domains of fixed area is attained on the disc. The rigorous proof was given by Faber (in 2D) [**28**] and Krahn (in full generality) [**66**]: the minimum of the first Dirichlet eigenvalue among all domains in $\mathbb{R}^n$ of fixed volume is attained on the ball.

A similar question can be asked for the spectrum of the closed problem, the Neumann spectrum and the Steklov spectrum. In this thesis we do not focus on the geometric optimization of Nemannn eigenvalues. We only notice that the volume constraint also makes the problem scale invariant. However, instead of minimization of eigenvalues one considers their maximization.

From now on let us focus on the Laplace spectrum of a closed manifold and the Steklov spectrum. Let us consider the spectrum of the closed problem first. We consider $\lambda_k(g)$ as a functional on the set $\mathcal{R}(M)$ of Riemannian metrics on M:

$$\lambda_k \colon \mathcal{R}(M) \to \mathbb{R}_{\geqslant 0},$$
$$g \to \lambda_k(g).$$

However, it is easy to check that for a homothety $g \to tg$, where t is a positive real number, one has

$$\lambda_k(tg) = \frac{\lambda_k(g)}{t},$$

i.e. one can make λ_k arbitrarily big and arbitrarily small and the question of geometric optimization of λ_k becomes trivial. Following the idea of Rayleigh we consider the following scale invariant functional on $\mathcal{R}(M)$:

$$\overline{\lambda}_k(M,g) = \lambda_k(g) Vol(M{,}g)^{2/\dim M},$$

where $Vol(M{,}g)$ stands for the volume of the metric g on M. This functional is called the $k-$th *normalized* eigenvalue. Let us concentrate on its optimization. It turns out that one can always construct a sequence of metrics $\{g_n\}$ on M such that $\overline{\lambda}_k(M,g_n) \to 0$ as $n \to \infty$. Therefore, the question of minimization of $\overline{\lambda}_k(M,g)$ is trivial. What kind of optimization should we consider then? The following theorem enables us to answer this question.

Theorem 0.0.1 ([**65, 12, 43**]). *One has the following bounds*

- *If* $\dim M = 2$*, then there exists a constant* $C > 0$ *depending only on the topology of* M *such that*
$$\overline{\lambda}_k(M{,}g) \leqslant Ck.$$
- *If* $\dim M \geqslant 3$*, then the functional* $\overline{\lambda}_k(M{,}g)$ *is not bounded from above on the set* $\mathcal{R}(M)$*.*
- *In any dimension there exists a constant* $C([g]) > 0$ *depending only on the conformal class* $[g] = \{e^{2\omega}g|\ \omega \in C^\infty(M)\}$ *such that for every metric* $\tilde{g} \in [g]$ *one has*
$$\overline{\lambda}_k(M,\tilde{g}) \leqslant Ck^{\frac{2}{\dim M}}.$$

The first and the third results were proved by Korevaar and later by Hassannezhad with some improvements. The second result was proved by Colbois and Dodziuk.

Theorem 1.1 guarantees that the following quantities are well-defined $\Lambda_k(M) := \sup_{g\in\mathcal{R}(M)}\overline{\lambda}_k(M,g)$, if $\dim M = 2$ and $\Lambda_k(M,[g]) := \sup_{\tilde{g}\in[g]}\overline{\lambda}_k(M,\tilde{g})$, for any dimension. The connection between these two functionals is expressed by the formula

$$\Lambda_k(M) = \sup_{[g]}\Lambda_k(M,[g]).$$

We will consider the functional $\Lambda_k(M,[g])$ which is called the $k-$th *conformal eigenvalue* later on and now let us concentrate on the functional $\Lambda_k(M)$. We will say that a Riemannian metric g is *maximal* if $\overline{\lambda}_k(M,g) = \Lambda_k(M)$. Finding maximal metrics is a challenging problem. This problem is closely related to the theory of minimal submanifolds in the spheres with standard metric (see Section 0.0.2).

The list of known maximal metrics is rather short. The first result is due to Hersch [**44**]. He proved that the canonical round metric g_{can} on the sphere is the unique (up to a homothety) maximal metric for the functional $\overline{\lambda}_1(\mathbb{S}^2,g)$, i.e.

$$\Lambda_1(\mathbb{S}^2) = \overline{\lambda}_1(\mathbb{S}^2,g_{can}) = 8\pi.$$

A similar result about the projective plane is due to Li and Yau [**70**] who proved that the the metric on $\mathbb{RP}^2$ induced from the metric g_{can} on $\mathbb{S}^2$ by the antipodal identification is the unique (up to a homothety) maximal metric for the functional $\overline{\lambda}_1(\mathbb{RP}^2,g)$, i.e.

$$\Lambda_1(\mathbb{RP}^2) = \overline{\lambda}_1(\mathbb{RP}^2,g_{can}) = 12\pi.$$

Later Nadirashvili in [**81**] and Petrides in [**96**] showed that $\Lambda_2(\mathbb{S}^2) = 16\pi$ but there is no maximal metric. The value 16π is attained in the limit by a sequence of metrics degenerating to a union of 2 touching identical round spheres (kissing spheres). Further Nadirashvili and Sire showed in the paper [**86**] that a similar result holds for the functional $\overline{\lambda}_3(\mathbb{S}^2,g)$: $\Lambda_3(\mathbb{S}^2) = 24\pi$, there is no maximal metric and this value is attained in the limit by a sequence of metrics degenerating to a union of 3 touching identical round spheres. A similar result for the functional $\overline{\lambda}_2(\mathbb{RP}^2,g)$ was obtained by Nadirashvili and Penskoi in the paper [**88**]: $\Lambda_2(\mathbb{RP}^2) = 20\pi$, there is no maximal metric and the value 20π is attained in the limit by a sequence of metrics degenerating to a union of a standard $\mathbb{RP}^2$ and a standard $\mathbb{S}^2$ touching each other such that the ratio of their areas is 3:2. Very recently these results were generalized to the functionals $\overline{\lambda}_k(\mathbb{S}^2,g)$ and $\overline{\lambda}_k(\mathbb{RP}^2,g)$ for all k. Particularly, Karpukhin, Nadirashvili, Penskoi and Polterovich in [**56**] proved that $\Lambda_k(\mathbb{S}^2) = 8\pi k$, there is no maximal metric whenever $k \geqslant 2$. The value $8\pi k$ is attained in the limit by a sequence of metrics degenerating to a union of k touching identical round spheres (a necklace of k spheres). Similarly, Karpukhin in [**53**] proved that $\Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$, there is no maximal metric whenever $k \geqslant 2$ and the value $4\pi(2k+1)$ is attained in the limit by a sequence of metrics degenerating to a union of the standard projective plane touching $k-1$ identical round spheres (a necklace of $\mathbb{RP}^2$ and $k-1$ spheres) such that the ratio of areas of $\mathbb{RP}^2$ and $\mathbb{S}^2$ is 3:2. Note that these results make use of a deep connection with algebraic geometry.

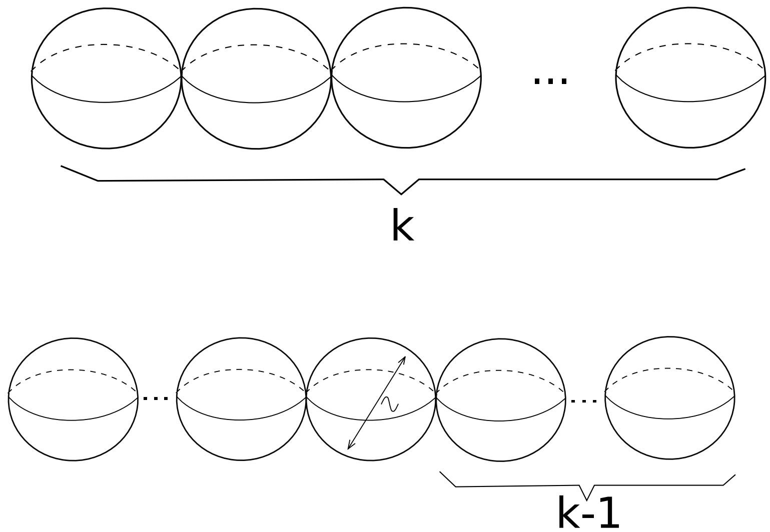

Fig. 1. Necklaces of k spheres (top) and of $\mathbb{RP}^2$ and $k-1$ spheres (bottom).

Further, let $\mathbb{T}^2$ and $\mathbb{KL}$ denote the 2-torus and the Klein bottle respectively. Nadirashvili [**80**] (see also [**37**]) proved that the unique (up to a homothety) maximal metric for the functional $\overline{\lambda}_1(\mathbb{T}^2, g)$ is the metric g_{eq} on the equilateral torus i.e. the torus obtained by taking quotient of $\mathbb{R}^2$ by the lattice composed of two equal equilateral triangles. Whence,

$$\Lambda_1(\mathbb{T}^2) = \overline{\lambda}_1(\mathbb{T}^2, g_{eq}) = \frac{8\pi^2}{\sqrt{3}}.$$

Nadirashvili also outlined a proof of the existence of maximal metrics for the functional $\overline{\lambda}_1(\mathbb{KL}, g)$. Later, in the paper [**47**] Jakobson, Nadirashvili and Polterovich found a candidate of a Riemannian metric on $\mathbb{KL}$ which could be a maximal metric. It was the metric on the Lawson bipolar surface $\tilde{\tau}_{3,1}$. It is a metric of revolution on $\mathbb{KL}$ which is defined by the formula

$$g_{\tilde{\tau}_{3,1}} = \frac{9 + (1 + 8\cos^2 v)^2}{1 + 8\cos^2 v}\left(du^2 + \frac{dv^2}{1 + 8\cos^2 v}\right),$$

where $0 \leqslant u < \pi/2, 0 < v \leqslant \pi$ and the corresponding first normalized eigenvalue is

$$\overline{\lambda}_1(\mathbb{KL}, g_{\tilde{\tau}_{3,1}}) = 12\pi E(2\sqrt{2}/3) \approx 13.365\pi,$$

where $E(\cdot)$ is a complete elliptic integral of the second kind. Finally, El Soufi, Giacomini and Jazar in [**21**] completed the proof that the metric $g_{\tilde{\tau}_{3,1}}$ is the unique (up to a homothety) maximal metric for $\overline{\lambda}_1(\mathbb{KL}, g)$. However, in all these proofs the only case of Riemannian metrics was considered. At the same time, it is known that the maximal metrics can have *conical singularities*. Geometrically in a small neighbourhood of a conical point these metrics conformal to Euclidean cones. They are not Riemannian metrics anymore since they vanish at conical singularities. We refer to this type of metrics as *metrics with conical singularities*.

The first examples of metrics with conical singularities which could be maximal were found on the orientable surface of genus 2 (denoted by Σ_2) by Jakobson, Levitin, Nadirashvili, Nigam and Polterovich in the paper [**46**]. It is metrics on *the Bolza surfaces*. These surfaces that we denote by $\mathcal{B}_\theta$ can be realized as (see [**89**])

$$\mathcal{B}_\theta = \{(z,w) \in \mathbb{C}^2 \mid w^2 = z(z^4 + 2z^2\cos 2\theta + 1)\} \cup \{\infty,\infty\},$$

where $0.65 \leqslant \theta \leqslant \pi/2 - 0.65$. Moreover, there exists a branched covering $\Pi_\theta \colon \mathcal{B}_\theta \ni (z,w) \to z \in \overline{\mathbb{C}^2} = \mathbb{S}^2$ with 6 ramification points. Then the metrics g_θ is the pullback by Π_θ of the round metric g_{can} on the sphere. Recently, in [**89**] Nayatani and Shoda proved that the metrics g_θ are indeed maximal. The corresponding first normalized eigenvalue is:

$$\Lambda_1(\Sigma_2) = \overline{\lambda}_1(\mathcal{B}_\theta, g_\theta) = 16\pi.$$

Finally, Matthiesen and Siffert in [**73**] showed that the maximum of the functional $\overline{\lambda}_1(\Sigma, g)$ is always attained on a metric with conical singularities. Thus, we need to extend the admissible set of metrics to the set of metrics with conical singularities. It motivated me and my co-authors D. Cianci and M. Karpukhin to check that the founded maximal metrics for the functionals $\overline{\lambda}_1(\mathbb{T}^2, g)$ and $\overline{\lambda}_1(\mathbb{KL}, g)$ still remain maximal if we allow the metric to have conical singularities. It was done in the paper [**11**]. Our result is

Theorem 0.0.2 (D.Cianci, M.Karpukhin, V.Medvedev [**11**])**.** *The maximal values $\Lambda_1(\mathbb{T}^2)$ and $\Lambda_1(\mathbb{KL})$ are achieved by smooth Riemannian metrics.*

There are two ingredients in the proof of Theorem 0.0.2. The first one is that the functional $\Lambda_1(\Sigma, [g])$ is continuous on $[g]$ with respect to *the Teichmüller distance*. It enables us to prove that the value $\Lambda_1(\Sigma)$ is always attained in a conformal class $[g]$ if $\Sigma = \mathbb{T}^2$ or $\mathbb{KL}$. Further, we use a result of Petrides who showed in [**96**] that the value $\Lambda_1(\Sigma, [g])$ is always attained on a metric with conical singularities. Whence we get that there is a metric g with conical singularities such that $\overline{\lambda}_1(\Sigma, g) = \Lambda_1(\Sigma)$ where Σ is either $\mathbb{T}^2$ or $\mathbb{KL}$. To prove that this metric admits no conical singularities we consider the $\mathbb{S}^1-$action on $\mathbb{T}^2$ and $\mathbb{KL}$ by conformal automorphisms. Our aim was to show that the metric g does not change under this action. In other words, g is a metric of revolution. Then if g has a conical singularity then it must admit a one-dimensional singular set of conical singularities which is impossible on a compact surface. To show that the metric g does not change under the action by conformal automorphisms we must know that in the conformal class of the metric g there is a unique metric induced from a branched minimal immersion into a round sphere by first eigenfunctions. It is the second ingredient in the proof of Theorem 0.0.2. Let us discuss it in more detail.

0.0.2. Branched minimal immersion of surfaces by first eigenfunctions

As we have noticed there exists a deep connection between minimal immersions into round spheres and maximal metrics for the functional $\overline{\lambda}_k(M,g)$. Recall that the immersion $\varphi\colon (M,\varphi^*h) \looparrowright (N,h)$ is called minimal if it is a critical point of the *volume functional* defined as:

$$V[\varphi] = \int_M dv_{\varphi^*h}.$$

The map φ is called a *branched* immersion if it is an immersion except for points $\{x_i\}$ where $d_{x_i}\varphi = 0$.

Theorem 0.0.3 (N.Nadirashvili, A.El Soufi, S. Ilias [**80, 24**])**.** *The maximal metrics for the functional $\overline{\lambda}_k(M,g)$ are among the metrics induced from branched minimal immersions by $k-$th eigenfunctions into a round sphere.*

Therefore, the branched minimal immersions become the main tool in the study of maximal metrics. Note that metrics induced from g_{can} on $\mathbb{S}^n$ by a branched minimal immersion have conical singularities at branched points.

As we also notice in the previous section it is important to know how many metrics induced from a minimal immersion by eigenfunctions into a sphere in a given conformal class. Montiel and Ros in [**78**] were the first who asked this question. Their result in the case of surfaces can be formulated as

Theorem 0.0.4 (S.Montiel, A.Ros [**78**])**.** *For each conformal class on a closed surface, there exists at most one metric which admits an isometric minimal immersion into a round sphere by first eigenfunctions.*

However, as we have also shown in the previous section finding maximal metrics naturally requires extending the set of Riemannian metrics and the set of minimal immersions to the set of metrics with conical singularities and the set of branched minimal immersions respectively. This extension was a necessary step in the proof of Theorem 0.0.2. Our result is as follows.

Theorem 0.0.5 (D.Cianci, M.Karpukhin, V.Medvedev [**11**])**.** *Let Σ be a closed surface endowed with a conformal class c. Then c belongs to exactly one of the following categories:*

1) *There does not exist $g \in c$ such that g admits a branched minimal immersion to a sphere by first eigenfunctions;*
2) *There exists a unique $g \in c$ such that g admits a branched minimal immersion by first eigenfunctions to $\mathbb{S}^m$ whose image is not an equatorial 2-sphere;*
3) *There exists $g \in c$ such that g admits a branched minimal immersion by first eigenfunctions to $\mathbb{S}^2$. In this case any two such immersions differ by a post-composition with a conformal automorphism of $\mathbb{S}^2$.*

The third category of conformal classes is new. In [**11**, Example 2] we provide an example of a conformal class of the third category on the surface of genus 2: there are infinitely many non-isometric metrics in the conformal class $[g_\theta]$ of the metric on the Bolza surface $\mathcal{B}_\theta$ admitting a branched minimal immersion by first eigenfunctions to $\mathbb{S}^2$.

Theorem 0.0.5 can be considered as an application of spectral geometry to conformal geometry. Note that the conformal spectrum $\Lambda_k(\Sigma,[g])$ is an invariant of the conformal class $[g]$ on Σ. Therefore, the study of the conformal spectrum can be considered as another application to conformal geometry. Similarly, the quantity $\Lambda_k(\Sigma)$ is a topological invariant of Σ and the study of $\Lambda_k(\Sigma)$ is an application of spectral geometry to topology. Let us consider another interesting topological invariant called the *Friedlander-Nadirashvili invariant*.

0.0.3. The Friedlander-Nadirashvili invariants

Let us remind the relation between the functionals $\Lambda_k(M)$ and $\Lambda_k(M,[g])$:

$$\Lambda_k(M) = \sup_{[g]} \Lambda_k(M,[g]).$$

Of course, this relation makes sense while $\dim M = 2$. The $k-$th Friedlander-Nadirashvili invariant denoted by $I_k(M)$ is defined in a similar way but we take the infimum in place of the supremum. Precisely,

$$I_k(M) := \inf_{[g]} \Lambda_k(M,[g]).$$

Note that unlike the functional $\Lambda_k(M)$ the Friedlander-Nadirashvili invariant is defined in any dimension. A priori the Friedlander-Nadirashvili invariant is an invariant of the differential structure. This invariant is not trivial. It has been shown by Nadirashvili and Friedlander in the paper [**34**] in the case $k = 1$. They established that $I_1(M) \geqslant I_1(\mathbb{S}^n)$. The value $I_1(\mathbb{S}^n)$ is known. It is nothing but $\overline{\lambda}_1(\mathbb{S}^n, g_{can})$. The general result about $I_k(M)$ follows from the paper [**13**] by Colbois and El Soufi who studied the conformal spectrum $\Lambda_k(M,[g])$: $I_k(M)$ is also non-trivial and $I_k(M) \geqslant I_k(\mathbb{S}^n)$. Nothing more is known in the case of $\dim M > 2$. Let us pass to the case of $\dim M = 2$.

The first result that we get for free is the values of $I_k(\mathbb{S}^2)$ and $I_k(\mathbb{RP}^2)$: $I_k(\mathbb{S}^2) = \Lambda_k(\mathbb{S}^2) = 8\pi k$ and $I_k(\mathbb{RP}^2) = \Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$ since any two metrics on $\mathbb{S}^2$ or $\mathbb{RP}^2$ are conformally equivalent.

Friedlander and Nadirashvili in [**34**] conjectured that $I_1(M) = 8\pi$ for any closed surface M other than the projective plane. This conjecture was confirmed in certain cases. So in the paper [**37**] Girouard proved that $I_1(\mathbb{KL}) = I_1(\mathbb{T}^2) = I_1(\mathbb{S}^2) = 8\pi$ (see also [**80**]). Later in the paper [**96**] Petrides extended the proof of Girouard. He proved that if M is a smooth compact *orientable* surface then $I_1(M) = 8\pi$ and the infimum is attained only on the sphere $\mathbb{S}^2$.

In the work [**55**] joint with M. Karpukhin we proved the following theorem.

Theorem 0.0.6 (M.Karpukhin, V.Medvedev [**55**]). *The following statements hold.*

(i) The Friedlander-Nadirashvili invariants of an orientable surface Σ_γ of genus γ satisfy $I_k(\Sigma_\gamma) = I_k(\mathbb{S}^2) = 8\pi k$ for any $\gamma \geqslant 0$. The infimum is attained if and only if $\gamma = 0$.

(ii) The Friedlander-Nadirahsvili invariants of a non-orientable surface Σ_γ of odd genus $\gamma \geqslant 1$ satisfy $I_k(\Sigma_\gamma) = I_k(\mathbb{S}^2) = 8\pi k$. The infimum is never attained.

(iii) The Friedlander-Nadirashvili invariants of a non-orientable surface Σ_γ of even genus $\gamma \geqslant 2$ satisfy

$$I_k(\Sigma_\gamma) \leqslant I_k(\Sigma_{\gamma-2}).$$

If this inequality is strict, then there exists a conformal class c such that $I_k(\Sigma_\gamma) = \Lambda_k(\Sigma_\gamma, c)$.

The proof of this theorem combines geometric and analytic techniques. The geometric part makes use of the theory of moduli spaces of conformal structures on a given closed surface. The main idea of this part is to investigate the behaviour of the quantity $\Lambda_k(M, c_n)$ when the sequence of conformal classes $\{c_n\}$ escapes to infinity in the moduli space of conformal classes on M. The analytic part relies on the theory of PDE's. The main technical tool here is the discontinuous metrics of type ρg, where g is a smooth Riemannian metric and ρ is a discontinuous positive function.

As a corollary of Theorem 0.0.6 we get

Corollary 0.0.7 (M.Karpukhin, V.Medvedev [**55**]). *If $\gamma \geqslant 2$ is even, then one has*

$$8\pi k = I_k(\mathbb{S}^2) < I_k(\Sigma_\gamma) \leqslant I_k(\mathbb{RP}^2).$$

In particular, for $k = 1$ one has

$$8\pi < I_1(\Sigma_\gamma) \leqslant 12\pi,$$

for all even γ.

Therefore, Corollary 0.0.7 shows that the statement "$I_1(M) = 8\pi$ unless M is a projective plane" suggested by Friedlander and Nadirashvili does not hold for non-orientable surfaces of even genus. Note that the exact value of $I_k(\Sigma_\gamma)$ remains unknown. In [**55**] we conjecture that:

Conjecture 0.0.8 (M.Karpukhin, V.Medvedev [**55**]). *For non-orientable closed surfaces Σ_γ of even genus γ one has $I_k(\Sigma_\gamma) = I_k(\mathbb{RP}^2)$. The infimum is attained if and only if $\gamma = 0$.*

One can ask the following question: why do the quantities $I_k(\gamma)$ take different values for odd and even γ? We suppose that the answer lies in the theory of *cobordisms*. Recall that two closed manifolds M and M' of the same dimension are called *cobordant* if there exists a manifold with boundary W such that the boundary ∂W is the disjoint union $M \sqcup M'$. Similarly, M is cobordant to 0 or null cobordant if there exists W such that $\partial W = M$. All the orientable compact surfaces can be realized as boundaries of $3-$dimensional compact manifold with boundary. For example the sphere is the boundary of the ball, the torus is the boundary of the solid torus etc. Similarly, the Klein bottle is the boundary of the solid Klein bottle. Therefore, all the orientable surfaces as well as the Klein bottle are cobordant to 0. It is well-known that two manifolds are cobordant if and only if they can be obtained from one another by a sequence of surgeries. In dimension 2 it implies that attaching a handle does not change the cobordism class. Hence all the non-orientable surfaces of odd genus are cobordant to the Klein bottle and all the non-orientable surfaces of even genus are cobordant to $\mathbb{RP}^2$. The fact that $\mathbb{RP}^2$ is not cobordant to 0 can be shown using Stiefel-Whitney characteristic classes. In [**55**] we conjecture that this phenomenon happens in any dimension. Precisely,

Conjecture 0.0.9 (M.Karpukhin, V.Medvedev [**55**])**.** *The quantities I_k are cobordism invariants, i.e. if M is cobordant to M' then $I_k(M) = I_k(M')$. In particular, if M is cobordant to* 0 *then $I_k(M) = I_k(\mathbb{S}^{\dim M}) = \Lambda_k(\mathbb{S}^{\dim M},[g_{can}])$.*

0.0.4. Spectral geometry of the Steklov problem

In this section we consider the Steklov problem (0.0.3) on compact surfaces with boundary. Introduced in the early 20th century, this problem is now becoming an increasingly popular object of attention of various research. Particularly, recently the Steklov problem has found applications in such areas of science as computer science (precisely, geometry processing and shape analysis, see [**106**]) and tomography (see [**41**] for the details). Note also that there exist applications to hydrodynamics (the so-called *sloshing problem*, see [**69**]).

Let (Σ, g) be a compact Riemannian surface and $\partial\Sigma$ its boundary. Recall that we denote the $k-$th Steklov eigenvalue by $\sigma_k(g)$. The first question of geometric optimization of $\sigma_k(g)$ goes back to the classical research of Weinstock on planar domains [**107**]. Similarly to the closed problem we shall consider $\sigma_k(g)$ as a functional on the set $\mathcal{R}(\Sigma)$ of Riemannan metrics on Σ

$$\sigma_k \colon \mathcal{R}(\Sigma) \to \mathbb{R}_{\geqslant 0},$$
$$g \to \sigma_k(g).$$

Considering homotheties $g \to tg$, implies that

$$\sigma_k(tg) = \frac{\sigma_k(g)}{\sqrt{t}},$$

and we see that one can make σ_k arbitrarily big and arbitrarily small. So the question of geometric optimization of σ_k becomes trivial. As in the closed problem we introduce the $k-$th *normalized* Steklov eigenvalue

$$\overline{\sigma}_k(\Sigma, g) = \sigma_k(g)L(\partial\Sigma,g),$$

where $L(\partial\Sigma,g)$ denotes for the length of the boundary $\partial\Sigma$ in the metric g. This functional is scale invariant. However, it turns out that for every surface one can build a sequence of metrics $\{g_n\}$ such that $\overline{\sigma}_k(\Sigma,g) \to 0$ as $n \to \infty$. Therefore, the question of the infimum of this functional is trivial. It was shown in [**16**] (see also [**43, 63**]) that if Σ is an orientable surface then the functional $\overline{\sigma}_k(\Sigma,g)$ is bounded from above. Moreover, the following theorem holds

Theorem 0.0.10 (A.Girouard, I.Polterovich [**39**])**.** *Let (Σ,g) be a compact orientable surface of genus γ with l boundary components. Then one has*

$$\overline{\sigma}_k(\Sigma,g) \leqslant 2\pi k(\gamma + l).$$

Surprisingly enough, a similar estimate was not known in the case of non-orientable compact surfaces with boundary. It has been found in the paper [**75**].

Theorem 0.0.11 (V.Medvedev [**75**])**.** *Let Σ be a compact non-orientable surface of genus γ with l boundary components. Then one has*

$$\overline{\sigma}_k(\Sigma,g) \leqslant 4\pi k(\gamma + 2l).$$

Here the genus of a non-orientable surface is defined as the genus of its orientable cover.

These theorems enable as to define the following quantities

$$\sigma_k^*(\Sigma) := \sup_{\mathcal{R}(\Sigma)} \overline{\sigma}_k(\Sigma,g),$$

and

$$\sigma_k^*(\Sigma,[g]) := \sup_{[g]} \overline{\sigma}_k(\Sigma,g),$$

called *the $k-$th conformal Steklov eigenvalue*. Therefore, $\sigma_k^*(\Sigma)$ is a topological invariant of Σ, while $\sigma_k^*(\Sigma,[g])$ is an invariant of the conformal class $[g]$ on Σ. Let's concentrate our attention on $\sigma_k^*(\Sigma,[g])$. Little is known about this functional. Since the disc has the unique conformal structure one can see that $\sigma_k^*(\mathbb{D}^2,[g_{can}]) = \sigma_k^*(\mathbb{D}^2)$, where g_{can} denotes the Euclidean metric on $\mathbb{D}^2$ with unit boundary length. Weinstock in the paper [**107**] found that $\sigma_1^*(\mathbb{D}^2) = 2\pi$. Later Girouard and Polterovich in [**40**] showed that $\sigma_k^*(\mathbb{D}^2) = 2\pi k$ for all $k \geqslant 1$. Conformal Steklov spectrum of a surface Σ was studied in [**99**] by Petrides. His result is

Theorem 0.0.12 (R.Petrides [**99**])**.** *For every Riemannian metric g on a compact surface Σ with boundary one has*

$$\sigma_k^*(\Sigma,[g]) \geqslant \sigma_{k-1}^*(\Sigma,[g]) + \sigma_1^*(\mathbb{D}^2,[g_{can}]), \tag{0.0.4}$$

particularly

$$\sigma_k^*(\Sigma,[g]) \geqslant 2\pi k. \tag{0.0.5}$$

Moreover, if the inequality 1.1 is strict then there exists a Riemannian metric $\tilde{g} \in [g]$ such that $\overline{\sigma}_k(\Sigma,\tilde{g}) = \sigma_k^*(\Sigma,[g])$.

New results about the functional $\sigma_k^*(\Sigma,[g])$ follow from the papers [**58, 38**]:

Theorem 0.0.13 ([**58, 38**])**.** *Let Σ_γ be a closed surface of genus γ and $\Sigma_{\gamma,l} \subset \Sigma_\gamma$ be a domain of genus γ with l boundary components in it. Then for any conformal class c on Σ_γ and for all k one has*

$$\lim_{l\to\infty} \sigma_k^*(\Sigma_{\gamma,l},c) \geqslant \Lambda_k(\Sigma_\gamma,c).$$

Moreover, if $k = 1$ or 2 one has

$$\sigma_k^*(\Sigma_{\gamma,l},c) \leqslant \Lambda_k(\Sigma_\gamma,c).$$

The first result is due to Girouard and Lagacé. The second result was proved by Karpukhin and Stern.

Let us consider the set of conformal classes on a given surface with boundary Σ. Recall that by the *Uniformization theorem* conformal classes on Σ are in one-to-one correspondence (up to an isometry) with metrics h on Σ of constant Gauss curvature and geodesic boundary. We introduce the C^∞ topology on the set of constant Gauss curvature and geodesic boundary. We get a geometric description of the *moduli space of conformal classes* on Σ. For any sequence of conformal classes $\{c_n\}$ then we can consider a sequence of the corresponding metrics $\{h_n\}$ of constant Gauss curvature and geodesic boundary. For any sequence $\{h_n\}$ there exist two possibilities: either the injectivity radii $inj(\Sigma,h_n) \to 0$ or $inj(\Sigma,h_n) \nrightarrow 0$ as $n \to \infty$. In the first case we get a genuine Riemannian surface (Σ,h_∞) with boundary in the limit. In the second case we say that the sequence of conformal classes $\{c_n\}$ *degenerates*. It turns out that in this case there exists a finite collection of pairwise disjoint geodesics for the metrics h_n whose lengths in h_n tend to 0 as n tends to ∞. We call these geodesics *pinching* or *collapsing*. They can be of the following three types: the collapsing boundary components, the collapsing geodesics with no self-intersection having two points of intersection with $\partial\Sigma$ and the collapsing geodesics with no self-intersection and which do not cross $\partial\Sigma$. In the limit as $n \to \infty$ we get a non-compact surface (Σ_∞,h_∞) with cusps. We compactify each cusp by a point and denote the obtained surface by $\widehat{\Sigma}_\infty$. Let $\widehat{h_\infty}$ denote the metric of constant Gauss curvature and geodesic boundary on $\widehat{\Sigma}_\infty$ such that $\widehat{h_\infty}_{|_{\Sigma_\infty}} = h_\infty$ and c_∞ its conformal class. In the paper [**75**] we establish the correspondence between $\sigma_k^*(\widehat{\Sigma}_\infty,c_\infty)$ and the limit of $\sigma_k^*(\Sigma,c_n)$ when the sequence of conformal classes c_n degenerates.

Theorem 0.0.14 (V.Medvedev [**75**])**.** *Let Σ be a compact surface of genus γ with $l > 0$ boundary components and let $c_n \to c_\infty$ be a degenerating sequence of conformal classes. Consider the corresponding sequence $\{h_n\}$ of metrics of constant Gauss curvature and geodesic boundary. Suppose that there exist s_1 collapsing boundary components and s_2 collapsing geodesics with no self-intersection which cross the boundary at two points. Moreover, suppose* *that $\widehat{\Sigma_\infty}$ has m connected components Σ_{γ_i,l_i} of genus γ_i with $l_i > 0$ boundary components, $\gamma_i + l_i < \gamma + l$, $i = 1, \ldots, m$. Then one has*

$$\lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = \max\Big(\sum_{i=1}^{m} \sigma_{k_i}^*(\Sigma_{\gamma_i,l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2)\Big), \tag{0.0.6}$$

where the maximum is taken over all possible combinations of indices such that

$$\sum_{i=1}^{m} k_i + \sum_{i=1}^{s_1+s_2} r_i = k.$$

Theorem 0.0.14 can be applied to the study of the question when the $k-$th conformal Steklov eigenvalue is attained on a metric in its conformal class.

Similarly to the closed problem one can introduce the following topological invariant of Σ :

$$I_k^\sigma(\Sigma) := \inf_{[g]} \sigma_k^*(\Sigma,[g]).$$

It is an analog of the Friedlander-Nadirashvili invariant for the Steklov problem. It is natural to ask what is the value of $I_k^\sigma(\Sigma)$ for a given surface Σ and what kind of topological invariant is $I_k^\sigma(\Sigma)$? The both questions were answered in [**75**] using Theorem 0.0.14. Our result is

Theorem 0.0.15 (V.Medvedev [**75**])**.** *Let Σ be a compact surface with boundary. Then one has $I_k^\sigma(\Sigma) = I_k^\sigma(\mathbb{D}^2) = 2\pi k$.*

Unlike the Friedlander-Nadirashvili invariant we may notice that I_k^σ are invariants of cobordisms of compact surfaces with boundary. Let us recall that two compact surfaces with boundary $(\Sigma_1,\partial\Sigma_1)$ and $(\Sigma_2,\partial\Sigma_2)$ are called cobordant if there exists a 3-dimensional *manifold with corners* Ω whose boundary is $\Sigma_1 \cup_{\partial\Sigma_1} W \cup_{\partial\Sigma_2} \Sigma_2$, where W is a cobordism of $\partial\Sigma_1$ and $\partial\Sigma_2$ (i.e. W is a surface with boundary $\partial\Sigma_1 \sqcup \partial\Sigma_2$). It turns out that the cobordisms of surfaces with boundary are trivial, i.e. all the surfaces with boundary are cobordant to 0. A fundamental fact about cobordisms of surfaces with boundary is *Theorem about splitting cobordisms* (see [**7**, Theorem 4.18]) which says that every cobordism of compact surfaces with boundary can be split into a sequence of cobordisms given by a handle attachment and cobordisms given by a *half-handle* attachment. In dimension 2 a half-handle attachment corresponds to attaching a strip along boundary components. Essentially in the proof of Theorem 0.0.6 we show that the value of I_k^σ does not change under handle and half-handle attachments. By this procedure any surface Σ can be reduced to the disc. Therefore, we get $I_k^\sigma(\Sigma) = I_k^\sigma(\mathbb{D}^2) = 2\pi k$.

0.0.5. Plan of the thesis

The first chapter, titled *On branched minimal immersions of surfaces by first eigenfunctions* [**11**], has been written in collaboration with Donato Cianci and Mikhail Karpukhin. Here we prove Theorems 0.0.2 and 0.0.5 (see Theorems 1.5 and 1.4 respectively).

The second chapter, titled *On the Friedlander-Nadirashvili invariants of surfaces* [**55**], has been written in collaboration with Mikhail Karpukhin. In this work we study the behaviour of the conformal spectrum on the moduli space of conformal classes of a given closed surface. This chapter contains the proofs of Theorem 0.0.6 (see Theorem 1.5) and Corollary 0.0.7 (see Corollary 1.6) as well as the discussion of Conjectures 0.0.8 and 0.0.9 (see Conjectures 1.9 and 1.10).

In the last chapter, titled *Degenerating sequences of conformal classes and the conformal Steklov spectrum* [**75**], we turn our attention to the conformal Steklov spectrum. Similarly to the second chapter our primary interest in this chapter is the behaviour of the conformal Steklov spectrum on the moduli space of conformal classes of a given compact surface with boundary when the sequence of conformal classes escapes to infinity. This chapter is dedicated to the proofs of Theorems 0.0.14 (see Theorem 1.7) and 0.0.15 (see Theorem 1.2).

First Chapter.

On branched minimal immersions of surfaces by first eigenfunctions

by

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Abstract. It was proved by Montiel and Ros that for each conformal structure on a compact surface there is at most one metric which admits a minimal immersion into some unit sphere by first eigenfunctions. We generalize this theorem to the setting of metrics with conical singularities induced from branched minimal immersions by first eigenfunctions into spheres. Our primary motivation is the fact that metrics realizing maxima of the first non-zero Laplace eigenvalue are induced by minimal branched immersions into spheres. In particular, we show that the properties of such metrics induced from $\mathbb{S}^2$ differ significantly from the properties of those induced from $\mathbb{S}^m$ with $m > 2$. This feature appears to be novel and needs to be taken into account in the existing proofs of the sharp upper bounds for the first non-zero eigenvalue of the Laplacian on the 2-torus and the Klein bottle. In the present paper we address this issue and give a detailed overview of the complete proofs of these upper bounds following the works of Nadirashvili, Jakobson-Nadirashvili-Polterovich, El Soufi-Giacomini-Jazar, Nadirashvili-Sire and Petrides.
Keywords: Spectral Theory, Branched minimal immersions, maximal metrics

1. Introduction

Let (Σ, g) denote a closed, connected Riemannian surface where the metric g is induced from a minimal isometric immersion into a round sphere of radius r. That is, $\Phi\colon (\Sigma,g) \to (\mathbb{S}^n_r, g_{\mathrm{can}})$ is a minimal isometric immersion. By a well known result of Takahashi [**103**, Theorem 3], the coordinate functions of such minimal immersions Φ are given by eigenfunctions for the Laplace-Beltrami operator on (Σ, g) with corresponding eigenvalue $\frac{2}{r^2}$. However, not all immersions are by *first* eigenfunctions. The following theorem shows that each conformal class of Σ admits at most one metric induced from an immersion into a sphere by first eigenfunctions:

Theorem 1.1 ([**22, 78**]). *For each conformal structure on a compact surface, there exists at most one metric which admits an isometric immersion into some unit sphere by first eigenfunctions.*

In this article we generalize Theorem 1.1 to the setting of *branched* minimal immersions into round spheres by first eigenfunctions (see Theorem 1.4 for a precise statement). Branched minimal immersions are given by smooth maps $\Phi\colon \Sigma \to \mathbb{S}^n$ which are minimal immersions except at finitely many points at which Φ becomes singular. In this situation, the pullback metric $\Phi^* g_{\mathrm{can}}$ on Σ will possess conical singularities at the singular points of Φ. Branched minimal immersions into spheres by first eigenfunctions occur in the study of metrics which maximize the first non-zero Laplace eigenvalue, denoted λ_1, among all metrics of area one. Indeed, in [**73**], Matthiesen and Siffert proved that for any closed surface Σ there exists a metric $\widehat{g}$ of area one, smooth except for possibly finitely many points which correspond to conical singularities, that maximizes λ_1 among all other unit-area metrics on Σ. These maximal metrics are induced from branched minimal immersions into a round sphere by first eigenfunctions and do in general possess conical singularities (see [**89**]). Therefore, it is natural to study Theorem 1.1 in the context of branched minimal immersions.

A technical difficulty unique to the branched immersion case is that one can have branched minimal immersions by first eigenfunctions whose images are an equatorial 2-sphere. Indeed, the conclusion of Theorem 1.1 is valid only with the restriction that the image of the branched minimal immersion is not an equatorial 2-sphere. This restriction indicates that the branched minimal immersions by first eigenfunctions into $\mathbb{S}^2$ are in a way special. Moreover, we show that if a conformal class has a metric induced by a branched minimal immersion by first eigenfunctions to $\mathbb{S}^2$ then it does not have a metric induced by a non-trivial branched minimal immersion by first eigenfunctions to a higher-dimensional sphere.

Theorem 1.1 has been applied to help classify certain metrics which maximize λ_1 (see the discussion in the next section). However, our generalization of Theorem 1.1 presents a novel feature that needs to be taken into account in this classification.

In the present article we address this issue by proving that there are no branched minimal immersions of a torus or a Klein bottle to $\mathbb{S}^2$. In order to precisely state our results, we give a more detailed version of the previous discussion.

1.1. Maximization of the first eigenvalue on surfaces and minimal immersions

After fixing a surface Σ, let $\mathcal{R}(\Sigma)$ be the collection of Riemannian metrics on Σ. We have the following homothety invariant functional on $\mathcal{R}(\Sigma)$:

$$\bar{\lambda}_1 : \mathcal{R}(\Sigma) \to \mathbb{R}_{\geq 0}; \qquad \bar{\lambda}_1(\Sigma, \cdot) : g \mapsto \lambda_1(g) vol(\Sigma, g),$$

where $\lambda_1(g)$ is the first non-zero Laplace-Beltrami eigenvalue of (Σ, g) and $vol(\Sigma,g)$ is the area of (Σ, g).

Using the notion of *conformal volume* (see Section 2), Li and Yau [**70**] established the following upper bound for $\bar{\lambda}_1(\Sigma,g)$ when Σ is orientable and has genus γ (see also [**109**]):

$$\bar{\lambda}_1(\Sigma,g) \leq 8\pi \left\lfloor \frac{\gamma+3}{2} \right\rfloor, \tag{1.1}$$

where the bracket denotes the integer part of the number inside. Modifying the ideas of Li and Yau, the second author [**51**, Theorem 1] proved the following upper bound for non-orientable surfaces (of genus γ):

$$\bar{\lambda}_1(\Sigma, g) \leq 16\pi \left\lfloor \frac{\gamma+3}{2} \right\rfloor. \tag{1.2}$$

Here the genus of a non-orientable surface is defined to be the genus of its orientable double cover.

Thus, $\bar{\lambda}_1(\Sigma,g)$ is bounded above on $\mathcal{R}(\Sigma)$. Naturally, one is interested in finding sharp upper bounds for $\bar{\lambda}_1(\Sigma, g)$ for a given surface and also characterizing the *maximal* metrics.

Definition 1.1. *Let Σ be a closed surface. A metric g_0 on Σ is said to be maximal for the functional $\bar{\lambda}_1(\Sigma,g)$ if*

$$\bar{\lambda}_1(\Sigma,g_0) = \sup_{g\in\mathcal{R}(\Sigma)} \bar{\lambda}_1(\Sigma,g).$$

Throughout, we will denote the value of $\sup_{g\in\mathcal{R}(\Sigma)} \bar{\lambda}_1(\Sigma,g)$ by $\Lambda_1(\Sigma)$. Additionally we set $\Lambda_1(\Sigma,[g])$ to be $\sup_{g\in[g]} \bar{\lambda}_1(\Sigma,g)$, where $[g]$ denotes the conformal class of a metric g. The following theorem guarantees the existence of a maximal metric on Σ, modulo finitely many points at which the metric may have conical singularities.

Theorem 1.2 ([**73**])**.** *For any closed surface Σ, there is a metric g on Σ, smooth away from finitely many conical singularities, achieving $\Lambda_1(\Sigma)$, i.e.*

$$\Lambda_1(\Sigma) = \bar{\lambda}_1(\Sigma,g).$$

Remark 1.2.
- *The proof of Theorem 1.2 uses results of Nadirashvili and Sire* [**83**] *and Petrides* [**96**] *on the maximization of $\Lambda_1(\Sigma,g)$ in a conformal class.*
- *Nayatani and Shoda* [**89**] *recently proved that Λ_1 is maximized by a metric on the Bolza surface with constant curvature one and six conical singularities (this metric was proposed to be maximal in* [**46**]*). Thus, Theorem 1.2 is optimal in regards to the regularity of a maximal metric.*

As the next theorem shows, these maximal metrics for $\bar{\lambda}_1$ are induced from branched minimal immersions into round spheres. It was first proved by Nadirashvili in [**80**] for the particular case of $\bar{\lambda}_1$. Later, the theorem was generalized to maximal metrics for higher Laplace eigenvalues in [**24**]. As noted in [**82**], the theorem also holds for metrics with conical singularities (in part because the variational characterization of λ_1 is the same whether considering metrics with conical singularities or smooth metrics). Together with Theorem 1.2, Theorem 1.3 is our motivation for studying branched minimal immersions by first eigenfunctions.

Theorem 1.3 ([**24, 63, 80**])**.** *Let g_0 be a metric on a closed surface Σ, possibly with conical singularities. Moreover, suppose that:*

$$\Lambda_1(\Sigma) = \bar{\lambda}_1(\Sigma, g_0).$$

Then g_0 is induced from a (possibly branched) minimal isometric immersion into a sphere by first eigenfunctions.

We briefly review some results regarding $\bar{\lambda}_1$-maximal metrics (for results regarding *extremal* metrics, see the survey [**94**] and the papers [**23, 59, 60, 61, 67, 91, 92, 93**]). By

Theorem 1.3, any $\bar{\lambda}_1$-maximal metric is induced by a (possibly branched) minimal immersion into a sphere. Hersch proved in 1970 that $\Lambda_1(\mathbb{S}^2)$ is achieved by any constant curvature metric [**44**]. By the work of Li and Yau [**70**], $\bar{\lambda}_1(\mathbb{RP}^2,g) \leq 12\pi$ for any metric g with equality for constant curvature metrics. Indeed, the metric of constant curvature one on $\mathbb{RP}^2$ can be realized as the induced metric from a minimal embedding into $\mathbb{S}^4$ called the Veronese embedding. Since there is only one conformal class of metrics on $\mathbb{RP}^2$, Theorem 1.1 shows that $\bar{\lambda}_1(\mathbb{RP}^2,g) \leq 12\pi$ with equality only if the metric is a constant curvature metric. In [**80**], Nadirashvili proved the existence of maximal metrics on the 2-torus (see also [**37**]) and outlined a proof of existence for metrics on the Klein bottle. In the next section we discuss the cases of the 2-torus and the Klein bottle in more detail. Finally, the maximal metric is known for Σ_2, the orientable surface of genus 2. Nayatani and Shoda proved in [**89**] that the metric on the Bolza surface proposed in [**46**] is maximal. As a result, $\Lambda_1(\Sigma_2) = 16\pi$.

1.2. Main results

We prove the following generalization of Theorem 1.1 to the setting of *branched* minimal immersions.

Theorem 1.4. *Let Σ be a closed surface endowed with a conformal class c. Then c belongs to exactly one of the following categories:*

1) *There does not exist $g \in c$ such that g admits a branched minimal immersion to a sphere by first eigenfunctions;*
2) *There exists a unique $g \in c$ such that g admits a branched minimal immersion by first eigenfunctions to $\mathbb{S}^m$ whose image is not an equatorial 2-sphere;*
3) *There exists $g \in c$ such that g admits a branched minimal immersion by first eigenfunctions to $\mathbb{S}^2$. Such metric g is not necessarily unique, but the corresponding immersions differ by a post-composition with a conformal automorphism of $\mathbb{S}^2$.*

Remark 1.3. *In Example 3.9 we provide a conformal class c of category 3) such that there exists a family of non-isometric metrics admitting a branched minimal immersion by first eigenfunctions to $\mathbb{S}^2$.*

Remark 1.4. *If Σ is not orientable, then $\Phi(\Sigma)$ can never be an equatorial 2-sphere. Indeed, this would make $\Phi\colon \Sigma \to \mathbb{S}^2$ a branched cover, which is impossible. In Proposition 3.11 we also prove that if Σ is a 2-torus the image of a branched minimal immersion by first eigenfunctions cannot be an equatorial 2-sphere. Thus, category 3) in Theorem 1.4 is not possible in these cases.*

Remark 1.5. *Theorem 1.4 allows us to construct an example where a maximal metric for $\bar{\lambda}_1$ cannot be induced by a branched minimal immersion whose components form a basis in the λ_1-eigenspace. Indeed, in* [**89**] *the authors showed that on a surface of genus 2 there exists a family of maximal metrics for $\bar{\lambda}_1$ induced from a branched minimal immersion to $\mathbb{S}^2$.*

Moreover, there are metrics in the family such that the multiplicity of the first eigenvalue is equal to 5*. At the same time by Theorem 1.4 such a metric can only be induced by a* 3*-dimensional family of eigenfunctions.*

The following theorem was proved by Nadirashvili in [**80**] for the case of $\mathbb{T}^2$, and the same paper contains an outline of the proof for the Klein bottle. Later, Girouard completed some of the steps of this outline in [**37**].

Theorem 1.5. *The maximal values* $\Lambda_1(\mathbb{T}^2)$ *and* $\Lambda_1(\mathbb{KL})$ *are achieved by smooth Riemannian metrics.*

Let us discuss the proof of Theorem 1.5 presented in [**80**]. Nadirashvili first shows that Theorem 1.2 holds for the torus, i.e. there exists a maximal metric possibly with conical singularities. Then he applies Theorem 1.1 to conclude that the maximal metric is flat. However, as we see from Theorem 1.4 the conclusion of Theorem 1.1 does not hold for branched immersions and special care is needed if the maximal metric happens to be induced by a branched minimal immersion to $\mathbb{S}^2$. Our contribution to Theorem 1.5 is that we show that there are no branched minimal immersions by first eigenfunctions to $\mathbb{S}^2$ from either the 2-torus or the Klein bottle. While the case of the Klein bottle is elementary, see Remark 1.4, additional considerations are required to settle the case of the 2-torus, see Proposition 3.11.

Once Theorem 1.5 is proved, Nadirashvili's argument shows that the maximal metric on $\mathbb{T}^2$ is flat. Let us recall it in more detail. It follows by Theorem 1.1 that any conformal transformation of a smooth maximal metric is an isometry. Since any metric on the 2-torus has a transitive group of conformal transformations, then any smooth maximal metric must have a transitive group of isometries and is therefore flat. It follows that a smooth maximal metric is a scalar multiple of the flat metric on the equilateral torus. In the same paper [**80**], Nadirashvili used a similar argument to deduce that any smooth maximal metric on the Klein bottle must be a surface of revolution. Later, Jakobson, Nadirashvili, and I. Polterovich [**47**] found a candidate for a smooth maximal metric for the Klein bottle (by proving the existence of a metric of revolution that was *extremal* for $\bar{\lambda}_1$). The metric they found corresponded to a bipolar surface of Lawson's $\tau_{3,1}$-torus. Then, in [**21**], El Soufi, Giacomini, and Jazar proved that this metric was the *only* smooth extremal metric on the Klein bottle.

Remark 1.6. *Note that Theorem 1.5 can be obtained by combining Theorem 1.4 and Proposition 3.11 with the recent work of Matthiesen and Siffert* [**73**]*. For completeness, we prove Theorem 1.5 without the result of Matthiesen and Siffert and instead combine results of Girouard* [**37**] *with the fact that the functional* $\Lambda_1(\Sigma,[g])$ *is continuous on the moduli space of conformal classes of metrics on* Σ*. The latter fact seems to be well-known but we were unable to find a reference. We present its proof in Section 4.*

As a result of the previous discussion we have the following corollaries of Theorem 1.5.

Corollary 1.6 ([**80**])**.** *The maximum for the functional* $\bar{\lambda}_1(\mathbb{T}^2,g)$ *on the space of Riemannian metrics on a 2-torus* $\mathbb{T}^2$ *is attained if and only if the metric* g *is homothetic to the flat metric* g_{eq} *on the equilateral torus and has the following value:*

$$\Lambda_1(\mathbb{T}^2) = 8\pi^2/\sqrt{3}.$$

Corollary 1.7 ([**21, 47**])**.** *The maximum for the functional* $\bar{\lambda}_1(\mathbb{KL},g)$ *on the space of Riemannian metrics on a Klein bottle* $\mathbb{KL}$ *is attained if and only if the metric* g *is homothetic to a metric of revolution:*

$$g_0 = \frac{9 + (1 + 8\cos^2 v)^2}{1 + 8\cos^2 v}\Big(du^2 + \frac{dv^2}{1 + 8\cos^2 v}\Big),$$

$0 \le u < \pi/2, 0 < v \le \pi$ *and has the following value:*

$$\Lambda_1(\mathbb{KL}) = 12\pi E(2\sqrt{2}/3) \approx 13.365\pi,$$

where $E(\cdot)$ *is a complete elliptic integral of the second kind.*

There are inconsistencies in the literature regarding whether there is a complete proof that the extremal metric on the Klein bottle found in [**47**] is indeed maximal. See, for instance, Remark 1.1 in [**21**]. One of the goals of the present article is to eliminate this inconsistency.

Paper outline. The rest of the paper is organized as follows: In Section 2 we provide the necessary background for studying branched minimal immersions into spheres and recall the definition of conformal volume. Section 3 contains the proofs of Theorem 1.4 and Proposition 3.11. In Section 4 we prove Theorem 1.5 and that the conformal spectrum is continuous on the moduli space of conformal classes of metrics on Σ.

2. Background

2.1. Branched immersions and conical singularities

Given a surface Σ endowed with a conformal structure one defines a metric g with conical singularities by declaring that at finitely many points $\{p_1,...,p_N\} \subset \Sigma$ (which are referred to as *conical points*) the metric has the following form in conformal coordinates centered at p_i: $\rho_i(z)|z|^{2\beta_i}|dz|^2$, where $\beta_i > -1$ and $\rho_i(z) > 0$ is smooth. The metric is singular in the sense that it becomes degenerate at the conical points. This approach is taken, for instance, in [**105**]. One can check that if $\rho = 1$ near the conical points then g is isometric to a cone with cone angle $2\pi(\beta_i + 1)$. In this article we are primarily concerned with metrics with conical singularities that arise from branched minimal immersions into spheres. A good introductory reference for branched minimal immersions is [**42**].

Fix a compact surface Σ equipped with a smooth Riemannian metric g_0 (without conical points). Let $\Phi\colon (\Sigma, g_0) \to (\mathbb{S}^n, g_{\text{can}})$ be a smooth map that is harmonic and conformal away from points at which $D_p\Phi = 0$. We will refer to points p at which $D_p\Phi = 0$ as *branch points* and call Φ a *branched conformal immersion*. Note that away from the branch points the quadratic form $g = \Phi^* g_{\text{can}}$ is actually an inner product on the tangent space and makes Φ a minimal immersion. Thus, we say that $\Phi\colon (\Sigma, g) \to (\mathbb{S}^n, g_{\text{can}})$ is a *branched minimal immersion* into a sphere. We will see that g possesses conical singularities at the branch points.

In a neighborhood of a branch point we can choose conformal coordinates $z = z_1 + iz_2$ on Σ centered at p and coordinates $x_1,...,x_n$ centered at $\Phi(p)$ such that $\Phi(z)$ takes the form:

$$x_1 + ix_2 = z^{m+1} + \sigma(z)$$

$$x_k = \phi_k(z); \;\; k \geq 3,$$

for $m \geq 1$ such that $\sigma(z)$ and $\phi_k(z)$ are $o(|z|^{m+1})$ and $\frac{\partial \sigma}{\partial z_j}$ and $\frac{\partial \phi_k}{\partial z_j}$ are $o(|z|^m)$ as $z \to 0$ (that this is possible follows from the discussion found in [**42**, Section 2]). The integer m is referred to as the *order* of the branch point. Moreover, there exist $C^{1,\alpha}$-coordinates (see Lemma 1.3 of [**42**]) $\tilde{z}$, which we will refer to as *distinguished parameters*, in which the map Φ takes the form:

$$x_1 + ix_2 = \tilde{z}^{m+1}$$

$$x_k = \psi_k(z); \;\; k \geq 3,$$

with $\psi_k(z)$ possessing the same asymptotics as $\phi_k(z)$ as $z \to 0$. The distinguished parameters are not an admissible coordinate system for the smooth structure on Σ since $\tilde{z}$ is related to z by $\tilde{z} = z\left[1 + z^{-(m+1)}\sigma(z)\right]^{1/(m+1)}$. By looking at the form Φ takes in distinguished parameters it is clear that $D_p\Phi \neq 0$ in a punctured neighborhood of a branch point. Thus, branch points are isolated. Moreover, since regular points form an open set, Σ can only posses finitely many branch points. From the previous discussion we see that in conformal coordinates centered at a branch point the metric is of the form $\rho(z)|z|^{2m}|dz|^2$, with $\rho(z) > 0$ smooth. In other words, near the branched point the metric is conformal to the Euclidean cone of total angle $2\pi(m+1)$. We will refer to m as the *order* of the conical singularity and will also refer to the branch points p as *conical points*.

We recall the following lemma, which allows one to define the tangent space to $\Phi(\Sigma)$ at the image of a conical point $\Phi(p)$. For simplicity, we state the lemma in the setting of branched conformal immersions into a round sphere. However, it holds in greater generality (see Lemma 3.1, 3.2 and the remark on page 771 of [**42**] for the proof).

Lemma 2.1. *Let $\Phi\colon \Sigma \to \mathbb{S}^n$ be a branched minimal immersion into a round sphere with a branch point at $p \in \Sigma$. Let w and x be distinguished parameters at p and $\Phi(p)$, respectively. Define the tangent space to $\Phi(p)$ in distinguished parameters as the x_1,x_2-plane.*

- *If $\{p_n\}$ is a sequence in Σ such that $p_n \to p$ and Φ is regular at p_n, then the tangent plane to $\Phi(\Sigma)$ at $\Phi(p_n)$ tends to the x_1,x_2-plane in distinguished parameters. Consequently, the Gauss map, which assigns to each point $q \in \Sigma$ the tangent plane to $\Phi(\Sigma)$ at $\Phi(q) \in \mathbb{S}^n$ is continuous on all of Σ.*
- *The definition of the tangent space to $\Phi(p)$ does not depend on the choice of distinguished parameters.*

Let g be a metric on Σ with conical singularities. Thus, $g = fg_0$, where g_0 is a smooth Riemannian metric on Σ and f is a smooth function on Σ that is nonzero except at possibly finitely many points. One can define the first Laplace eigenvalue corresponding to this singular metric using the variational characterization:

$$\lambda_1(g) = \inf_{\substack{u \in H^1(\Sigma,g)\\ u\perp 1}} R(u,g), \tag{2.1}$$

where:

$$R(u,g) = \frac{\int_\Sigma |\nabla u|^2 \mathrm{dV}(g_0)}{\int_\Sigma u^2 dvol(g)}$$

is the Rayleigh quotient and $H^1(\Sigma, g)$ is the completion of the set:

$$\left\{ u \in L^2(\Sigma, dvol(g)); \quad \int_\Sigma |\nabla u|^2 dvol(g_0) < \infty \right\}$$

with respect to the norm:

$$\|u\|^2_{H^1(g)} = \int_\Sigma u^2 dvol(g) + \int_\Sigma |\nabla u|^2 dvol(g_0).$$

When a metric g_0 is smooth, we will regard $H^1(\Sigma, g_0)$ as the usual Sobolev space. Note that, essentially by the conformal invariance of the Dirichlet energy, $H^1(\Sigma,g) = H^1(\Sigma,g_0)$, meaning that they are equal as sets, and the norms define the same topology (see [**105**, Proposition 3]). A function $u \in H^1(\Sigma, g)$ for which the infimum of the Rayleigh quotient in (2.1) is achieved is called *a first eigenfunction*. For metrics with conical singularities, first eigenfunctions exist (see [**63**] Proposition 1.3). By the usual elliptic regularity argument (see, for instance, [**36**, Corollary 8.11]), the first eigenfunctions are smooth and satisfy the following equation:

$$\Delta_{g_0} u = \lambda_1(g) f u.$$

Remark 2.2. *Similarly, the higher eigenvalues are defined by the standard variational characterisation (see for example* [**63**]*). The existence of corresponding eigenfunctions is guaranteed by* [**63**] *Proposition 1.3. The smoothness of the eigenfunctions is also a consequence of the elliptic regularity.*

Remark 2.3. *If one defines eigenvalues of the Laplacian as above, then Takahashi's theorem* [**103**, Theorem 3] *holds for surfaces equipped with metrics with conical singularities. The statement is as follows.*

Theorem 2.1. *Let M be a surface and g be a metric on M, possibly with conical singularities.*

- *Let $\Phi\colon (M,g) \to \mathbb{R}^{n+1}$ be a branched isometric immersion $\Phi = (\phi_1,\ldots,\phi_{n+1})$, where ϕ_i are eigenfunctions of the Laplacian Δ_g with the same eigenvalue λ. Then the image $\Phi(M)$ lies in the sphere $\mathbb{S}^n_R$ of radius $R = \sqrt{\frac{2}{\lambda}}$ and the map $\Phi\colon (M,g) \to \mathbb{S}^n_R$ is a branched minimal immersion.*
- *Conversely, let $\Phi\colon (M,g) \to \mathbb{S}^n_R \subset \mathbb{R}^{n+1}$ be a branched isometric minimal immersion into the sphere of radius R centered at the origin. If $\Phi = (\phi_1,\ldots,\phi_{n+1})$, then ϕ_i are eigenfunctions of Δ_g with the same eigenvalue $\lambda = \dfrac{2}{R^2}$.*

Proof. The original proof of Takahashi is purely local and, therefore, establishes the statement of the theorem in a neighbourhood of a regular point.

- We have $\sum_{i=1}^{n+1} \phi_i^2 \equiv R^2$ on the set of regular points and, therefore, everywhere by continuity. The map $\Phi\colon (M,g) \to \mathbb{S}^n_R$ is minimal in a neighbourhood of any regular point and is, therefore, a branched minimal immersion.
- Suppose that $g = fg_0$, where g_0 is a genuine Riemannian metric and f is a smooth non-negative function with zeroes at branch points. By definition Φ is an isometric minimal immersion at any regular point x. Hence, by the Takahashi's theorem one has $\Delta_{g_0}\phi_i(x) = \lambda f(x)\phi_i(x)$. The function $\Delta_{g_0}\phi_i - \lambda f\phi_i$ is smooth, globally defined and equals zero everywhere except at conical points. Thus, $\Delta_{g_0}\phi_i = \lambda_k f\phi_i$ at every point of Σ and ϕ_i are eigenfunctions with the same eigenvalue $\lambda = \dfrac{2}{R^2}$.

□

2.2. Conformal volume

The notion of conformal volume was introduced by Li and Yau to prove bounds on λ_1 that depend only on the genus [**70**]. It will be used in our poof of Theorem 1.4. Throughout, let G_n denote the group of conformal diffeomorphisms of the n-sphere with its canonical metric and let $\Phi\colon \Sigma \to \mathbb{S}^n$ be a conformal immersion with possible branch points.

Definition 2.4.

- *The conformal n-volume of Φ is given by:*

$$\mathrm{vol}_c(n,\Phi) := \sup_{\gamma\in G_n} \mathrm{vol}(\Sigma, (\gamma\circ\Phi)^* g_{\mathrm{can}}).$$

- *The conformal n-volume of Σ, denoted $\mathrm{vol}_c(n,\Sigma)$, is the infimum of $\mathrm{vol}_c(n,\Phi)$ over all branched conformal immersions $\Phi\colon \Sigma \to \mathbb{S}^n$.*

Remark 2.5. *In the recent preprint* [**64**] *Kokarev used conformal volume to obtain bounds for higher eigenvalues λ_k.*

3. Proofs of main results

To prove Theorem 1.4 we follow the same steps used by El Soufi and Ilias to prove the analog of Theorem 1.4 for minimal immersions without branch points (see Corollary 3.3 in [**22**]). While some propositions easily generalize to the setting of branched minimal immersions (compare Proposition 3.2 with [**22**, Theorem 2.2]) others do not generalize completely (compare Proposition 3.3 with [**22**, Proposition 3.1]).

Proposition 3.1. *Let (Σ, g) be a compact Riemannian surface possibly with conical singularities and suppose that there exists a branched minimal isometric immersion Φ of (Σ, g) into a round sphere of dimension n, then:*

$$\mathrm{vol}(\Sigma, g) = \mathrm{vol}_c(n, \Phi) \geq \mathrm{vol}_c(n, \Sigma).$$

Moreover, if $\Phi(\Sigma)$ is not an equatorial 2-sphere, then $\mathrm{vol}(\Sigma, g) > \mathrm{vol}(\Sigma, (\gamma \circ \Phi)^* g_{\mathrm{can}})$ *for every* $\gamma \in G \setminus \mathrm{O}(n+1)$.

Proof. Given a unit-vector $a \in \mathbb{R}^{n+1}$, let A denote the projection of the vector onto the tangent space of each point $\mathbb{S}^n$. Then A is the gradient vector field of the function $u = \langle \cdot, a \rangle$. Let $(\gamma_t^a)_t$ be the time-t map for the flow associated to A. Then $(\gamma_t^a)^* g_{\mathrm{can}} = e^{2f} g_{\mathrm{can}}$, with f a smooth function on the sphere. Recall that for every $\gamma \in G_n$ there exists $r \in \mathrm{O}(n+1)$ and γ_t^a such that $\gamma = r \circ \gamma_t^a$ (see the Lemma on page 259 of [**22**]). Thus, it suffices to show that:

$$\mathrm{vol}(\Sigma, (\gamma_t^a \circ \Phi)^* g_{\mathrm{can}}) \leq \mathrm{vol}(\Sigma, \Phi^* g_{\mathrm{can}}) \text{ for every } a \in \mathbb{S}^n \text{ and } t \geq 0.$$

First, we need to verify that we can make use of the first variation formula for the area of Σ. Let $\{p_1,...,p_N\}$ denote the branch points of Φ. Then

$$\widehat{\Sigma} := \Sigma \setminus \{p_1,...,p_N\}$$

is an (open) Riemannian manifold and Φ induces a minimal isometric immersion of $\widehat{\Sigma}$ into $\mathbb{S}^n$. For convenience, we will often identify $\widehat{\Sigma}$ and its image under Φ. In coordinates centered at a branch point of order m, the volume form is given by:

$$\mathrm{dV}((\gamma_t^a \circ \Phi)^* g_{\mathrm{can}})(z) = \rho(t,z)|z|^{2m}|dz \wedge d\bar{z}|,$$

where $\rho(t,z)$ is a smooth positive function. Thus, it is clear that the volume form is differentiable (smooth) in t and the derivative with respect to t is identically zero at the branch points. Set $\gamma = \gamma_{t_0}^a$. Away from the singular points, we have the usual expression for the derivative of the volume form:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=t_0} \mathrm{dV}((\gamma_t^a \circ \Phi)^* g_{\mathrm{can}})(x) = &- \left\langle A_{\gamma(x)}, H_{\gamma(x)}^{\gamma(\widehat{\Sigma})} \right\rangle \mathrm{dV}((\gamma \circ \Phi)^* g_{\mathrm{can}})(x) \\ &+ \mathrm{div}_{\gamma(\widehat{\Sigma})}(A_{\gamma(x)}^{\top}) dvol((\gamma \circ \Phi)^* g_{\mathrm{can}})(x), \end{aligned} \tag{3.1}$$

where $H^{\gamma\left(\widehat{\Sigma}\right)}_{\gamma(x)}$ is the mean curvature vector for $\gamma\left(\widehat{\Sigma}\right)$ and $A^{\top}$ is the projection of A onto the tangent space of $\gamma\left(\widehat{\Sigma}\right)$. Since Φ is minimal away from the branch points, one can compute the expression for the mean curvature vector explicitly away from the branch points (see page 260 of [**22**]):

$$H^{\gamma\left(\widehat{\Sigma}\right)}_{\gamma(x)} = -2e^{-2f} D\gamma((\nabla f^{\perp})_x).$$

Moreover, by Lemma 2.1, $\nabla f^{\perp}$ extends to a continuous vector field on all of $\Phi(\Sigma)$. Thus, $H^{\gamma(\Sigma)}$ extends to a continuous vector field on the branch points. It follows that the first term in the right hand side of (3.1) is zero at a branch point. Thus, the second term in the right hand side extends to something continuous and zero at the branch points.

Now we integrate both sides of (3.1) to recover the usual first variation formula. However, since $A^{\top}_x$ only extends to a continuous vector field on $\Phi(\Sigma)$, some care is required to show that the integral of the second term in the right hand side of (3.1) is zero. Notice that the integrals of the left hand side and the first term in the right hand side of (3.1) converge as improper integrals. Thus, it suffices to exhibit an exhaustion of $\gamma(\widehat{\Sigma})$ by compact sets with smooth boundary such that the integral of the second term on the right hand side of (3.1) converges to zero. Let $\{\Omega_n\}_{n=1}^{\infty}$ be an exhaustion of $\widehat{\Sigma}$ by compact sets with smooth boundary such that for each connected component of $\partial\Omega_n$ there exist distinguished parameters (z_1,z_2) centered at a singular point such that the image of the connected component of $\partial\Omega_n$ is given by $z_1^2 + z_2^2 = \frac{1}{n^2}$, for n large enough. Then $\{\gamma(\Omega_n)\}$ is an exhaustion of $\gamma(\widehat{\Sigma})$. The Divergence Theorem yields:

$$\begin{aligned}
\int_{\gamma(\Omega_n)} \mathrm{div}_{\gamma(\Omega_n)}\left(A^{\top}_x\right) \mathrm{dV}(g_{\mathrm{can}}) &= \int_{\partial\gamma(\Omega_n)} \langle A^{\top}_x, N_x\rangle \mathrm{d}s \\
&= \int_{\partial\Omega_n} \langle A^{\top}_{\gamma(x)}, N_{\gamma(x)}\rangle e^{2f} \mathrm{d}s \\
&= \int_{\partial\Omega_n} \langle D\gamma(A_x)^{\top}, N_{\gamma(x)}\rangle e^{2f} \mathrm{d}s \\
&= \int_{\partial\Omega_n} \langle D\gamma(A^{\top}_x), N_{\gamma(x)}\rangle e^{2f} \mathrm{d}s,
\end{aligned}$$

where N is the outward pointing unit normal vector field along $\partial\gamma(\Omega_n)$ and ds is the length element along $\partial\gamma(\Omega_n)$. Again by Lemma 2.1, $A^{\top}_x$ extends to a continuous vector field on $\Phi(\Sigma)$, then the Cauchy-Schwartz inequality shows that the integral is $O((1/n)^{2m+1})$. Thus, as an improper integral, we see that:

$$\int_{\Sigma} \mathrm{div}_{\gamma(\Sigma)}(A^{\top}_{\gamma(x)}) \mathrm{dV}((\gamma \circ \Phi)^* g_{\mathrm{can}}) = 0.$$

Integrating both sides of (3.1) yields the usual first variation formula:

$$\left.\frac{d}{dt}\right|_{t=t_0} \mathrm{vol}(\Sigma, (\gamma^a_t \circ \Phi)^* g_{\mathrm{can}}) = -\int_{\Sigma} \langle H^{\gamma(\widehat{\Sigma})}_{\gamma(x)}, A_{\gamma(x)}\rangle \;\, \mathrm{dV}\left((\gamma \circ \Phi)^* g_{\mathrm{can}}\right).$$

At this point, the calculation done in the proof of Theorem 1.1 of [**22**] applies:

$$\left.\frac{d}{dt}\right|_{t=t_0} \operatorname{vol}(\Sigma, (\gamma_t^a \circ \Phi)^* g_{\mathrm{can}}) = 2\int_\Sigma \frac{u - u\circ\gamma}{1-u^2} |A^\perp|^2 \ \mathrm{dV}((\gamma\circ\Phi)^* g_{\mathrm{can}}), \tag{3.2}$$

where the integrand is extended by continuity at the branch points. Since $u(x) \leq u(\gamma(x))$, we conclude that:

$$\left.\frac{d}{dt}\right|_{t=t_0} \operatorname{vol}(\gamma_t^a(\widehat{\Sigma})) \leq 0.$$

Thus, $\operatorname{vol}(\gamma_t^a(\Phi(\Sigma)))$ is non-increasing.

Now suppose that there exists a and $t_0 > 0$ such that $\operatorname{vol}(\gamma_{t_0}^a(\Phi(\Sigma))) = \operatorname{vol}(\Phi(\Sigma))$. Then

$$\left.\frac{d}{dt}\right|_{t=s} \operatorname{vol}(\gamma_t^a(\Phi(\Sigma))) = 0,$$

for $0 \leq s \leq t_0$. From this observation and (3.2) we conclude that $A^\perp = 0$ on $\Phi(\Sigma)$. Thus, A restricts to a vector field on $\widehat{\Sigma}$. Observe that the integral curves of A are great circles inside $\mathbb{S}^n$ connecting a and $-a$. Therefore, $a, -a \in \Phi(\Sigma)$. If a is a regular value of Φ, then $\Phi(\Sigma)$ is just given by the image of $T_a\Phi(\Sigma)$ under the Riemannian exponential map of $\mathbb{S}^n$ based at a. So $\Phi(\Sigma)$ is an equatorial 2-sphere.

Now suppose that a corresponds to a singular value of Φ. Again, by Lemma 2.1 we may define the tangent space to $\Phi(\Sigma)$ at a in $T_a\mathbb{S}^n$. Let V denote this subspace. Given $p \in \Phi(\widehat{\Sigma})$ sufficiently close to a, let $\alpha\colon [0,1] \to \mathbb{S}^n$ be the minimizing geodesic connecting p and a. Then $\alpha((0,1))$ is contained in $\Phi(\widehat{\Sigma})$. Moreover, since $\alpha'(t)$ is in the tangent space to $\Phi(\widehat{\Sigma})$ for every $t \in (0,1)$, then $\alpha'(1) \in V$. This shows that $\Phi(\Sigma)$ is again the image of V under the Riemannian exponential map of $\mathbb{S}^n$ at a. Thus, $\Phi(\Sigma)$ is a 2-sphere. □ □

The following proposition is a generalization of Theorem 2.2 in [**22**] to the setting of branched minimal immersions. See also Theorem 1 in [**70**]. The proof of Theorem 2.2 in [**22**] carries through without changes to the setting of branched minimal immersions.

Proposition 3.2. *Suppose (Σ, g) is a Riemannian surface with possible conical singularities. For all n such that the conformal n-volume is defined, we have:*

$$\bar{\lambda}_1(\Sigma, g) \leq 2\operatorname{vol}_c(n, \Sigma).$$

Equality holds if and only if (Σ, g) admits, up to homothety, a branched minimal immersion into a sphere by first eigenfunctions.

We generalize Proposition 3.1 of [**22**] to the setting of branched minimal immersions. However, the statement is complicated by the fact that the image of a branched minimal immersion can be an equatorial 2-sphere.

Proposition 3.3. *Let (Σ, g) be a surface with possible conical singularities. Moreover, suppose that the metric g is induced from a branched minimal immersion Φ into a sphere. Then* *every metric with possible conical singularities $\widetilde{g}$ conformal to g satisfies the following:*

$$\bar{\lambda}_1(\Sigma,\widetilde{g}) \leq 2\,\mathrm{vol}(\Sigma, g).$$

Criteria for equality are as follows:

- *If the image of Φ* is not *an equatorial 2-sphere, then equality holds if and only if the components of Φ are first eigenfunctions and $\widetilde{g}$ is homothetic to g.*
- *If the image of Φ* is *an equatorial 2-sphere, then equality holds if and only if there exists a conformal automorphism γ of $\mathbb{S}^2$ such that $\widetilde{g}$ is homothetic to $(\gamma \circ \Phi)^* g_{\mathrm{can}}$ and the components of $\gamma \circ \Phi$ are first eigenfunctions.*

Remark 3.4. *Note that in the second case the coordinates of Φ are not necessarily first eigenfunctions, see Example 3.7 below.*

Proof. Let $\widetilde{g} \in [g]$ be another metric with possible conical singularities from the conformal class of the metric g. Let Φ_i denote the i-th component function corresponding to Φ (as a map from Σ to $\mathbb{S}^n \subset \mathbb{R}^{n+1}$). According to the proof of Theorem 1 in [**70**], there exists a conformal automorphism of $\mathbb{S}^n$, denoted γ, such that for every i we have:

$$\int_\Sigma (\gamma \circ \Phi)_i \mathrm{dV}(\widetilde{g}) = 0. \tag{3.3}$$

Set $\Psi = \gamma\circ\Phi$. Then using the conformal invariance of the Dirichlet energy and the variational characterization of Laplace eigenvalues, we have:

$$\begin{aligned}
\lambda_1(\widetilde{g}) &\leq \frac{\sum_i \int_\Sigma |d\Psi_i|^2_g \,\mathrm{dV}(\widetilde{g})}{\sum_i \int_\Sigma \Psi_i^2 \,\mathrm{dV}(\widetilde{g})} && (3.4)\\
&= \frac{2\,\mathrm{vol}(\Sigma, (\gamma\circ\Phi)^* g_{\mathrm{can}})}{\mathrm{vol}(\Sigma, \widetilde{g})} && (3.5)\\
&\leq 2\,\mathrm{vol}(\Sigma, g)\,\mathrm{vol}(\Sigma, \widetilde{g})^{-1}, && (3.6)
\end{aligned}$$

where the second inequality follows from Proposition 3.1. The desired inequality follows.

Now assume that the equality is achieved, i.e. inequalities (3.4) and (3.6) turn into equalities and one has

$$\bar{\lambda}_1(\Sigma,\widetilde{g}) = 2\,\mathrm{vol}(\Sigma, g). \tag{3.7}$$

By Proposition 3.2 $\bar{\lambda}_1(\Sigma, \widetilde{g}) \leq 2\,\mathrm{vol}_c(n, \Sigma)$. Combining with equality (3.7), this yields

$$\mathrm{vol}(\Sigma, g) \leq \mathrm{vol}_c(n,\Sigma).$$

At the same time, by Proposition 3.1, the reverse inequality is true. Therefore, one has an equality.

Assume that the image is not an equatorial 2-sphere. Then by Proposition 3.1 γ is an isometry. Since any isometry of the sphere is linear, equality (3.3) is satisfied with $\gamma = \mathrm{id}$. Together with equality (3.4) this yields that coordinates of Φ are first eigenfunctions for the metric $\widetilde{g}$. If $\widetilde{g} = e^{2\omega} g$ then

$$\lambda_1(\widetilde{g})\Phi = \Delta_{\widetilde{g}}\Phi = e^{-2\omega}\Delta_g \Phi = e^{-2\omega}\lambda_1(g)\Phi. \tag{3.8}$$

We conclude that in this case ω is constant and $\widetilde{g}$ is homothetic to g.

Now suppose that the image is an equatorial 2-sphere. In this case we cannot conclude that γ is an isometry. Nevertheless, equalities in (3.3) and (3.4) imply that coordinates of Ψ are first eigenfunctions for $\widetilde{g}$. Setting $g' = \Psi^* g_{\text{can}}$ and $\widetilde{g} = e^{2\omega} g'$ we obtain similarly to (3.8),

$$\lambda_1(\widetilde{g})\Psi = \Delta_{\widetilde{g}}\Psi = e^{-2\omega}\Delta_{g'}\Psi = 2e^{-2\omega}\Psi.$$

We conclude that in this case ω is constant and $\widetilde{g}$ is homothetic to g'.

Let us prove the converse to the equality statements. If the components of Φ are first eigenfunctions then $\lambda_1(g) = 2$. Since $\tilde{g}$ is homothetic to g one has

$$\bar{\lambda}_1(\Sigma,\widetilde{g}) = \bar{\lambda}_1(\Sigma, g) = 2\operatorname{vol}(\Sigma,g).$$

Suppose that the image of Φ is an equatorial 2-sphere. Set $\Psi = \gamma \circ \Phi$ then after rescaling we may assume $\widetilde{g} = \Psi^* g_{\text{can}}$ and $\operatorname{vol}(\Sigma,g) = 4\pi|\deg \Phi| = 4\pi|\deg \Psi| = \operatorname{vol}(\Sigma,\widetilde{g})$, since conformal transformations preserve the absolute value of the degree. If the components of Ψ are first eigenfunctions then $\lambda_1(\tilde{g}) = 2$ and one has

$$\bar{\lambda}_1(\Sigma,\widetilde{g}) = 8\pi|\deg \Psi| = 8\pi|\deg \Phi| = 2\operatorname{vol}(\Sigma,g).$$

$\square$ $\square$

The following proposition is proved in [**79**, Theorem 6]. We reprove it here using a slightly different approach.

Proposition 3.5. *Suppose that $\Phi\colon \Sigma \to \mathbb{S}^2$ is a branched minimal immersion by first eigenfunctions. Then*

(i) For any other conformal map $\Psi\colon \Sigma \to \mathbb{S}^2$ one has $|\deg \Psi| \geq |\deg \Phi|$;

(ii) If $|\deg \Psi| = |\deg \Phi|$ then there exists a conformal transformation γ such that $\Psi = \gamma \circ \Phi$.

Proof. This proposition is a direct corollary of Proposition 3.3. To prove (i), we apply Proposition 3.3 for metrics $\widetilde{g} = \Phi^* g_{\text{can}}$ and $g = \Psi^* g_{\text{can}}$. Then $\lambda_1(\widetilde{g}) = 2$ and we conclude

$$8\pi|\deg \Phi| = \bar{\lambda}_1(\Sigma,\widetilde{g}) \leq 2\operatorname{vol}(\Sigma,g) = 8\pi|\deg \Psi|.$$

If $|\deg \Phi| = |\deg \Psi|$, then we switch the roles of g and $\widetilde{g}$ and observe that we have an equality, i.e. by Proposition 3.3 there exists a conformal automorphism γ_0 such that

$$(\gamma_0 \circ \Phi)^* g_{\text{can}} = \Psi^* g_{\text{can}}. \tag{3.9}$$

We would like to show that it implies the existence of an isometry I of $\mathbb{S}^2$ such that $\Psi = I \circ \gamma_0 \circ \Phi$.

Lemma 3.6. *Let f and h be two holomorphic maps $\Sigma \to \mathbb{S}^2$ (i.e. meromorphic functions) such that for any choice of local coordinates one has $|f_z| = |h_z|$. Then there exists $\alpha \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $f = e^{i\alpha}h + c$.*

Proof. First, note that the condition of the lemma implies that that f and h have the same singular sets. Let $p \in \Sigma$ be any regular point of f and h, i.e. $df(p) \neq 0$ and $dh(p) \neq 0$. Let z be local coordinates, then there exists a real-valued function $\alpha(z)$ such that $f_z = e^{i\alpha(z)}h_z$. Taking $\partial_{\bar z}$ of both parts we obtain

$$i(\partial_{\bar z}\alpha)e^{i\alpha}h_z = 0.$$

Since $h_z \neq 0$ in a neighborhood of p, one concludes that α is a real-valued holomorphic function. Thus, α is a constant. Integrating the equality $f_z = e^{i\alpha}h_z$, we obtain an equality $f = e^{i\alpha}h + c$ valid in a neighborhood of p. Since it is an equality between two meromorphic functions, by unique continuation it is valid everywhere on Σ. □ □

By taking conjugates if necessary, we can assume that $\gamma_0 \circ \Phi$ and Ψ are both holomorphic. Equality (3.9) guarantees that we can apply the previous lemma to these functions. The conclusion of the lemma then provides a desired isometry I. Setting $\gamma = I \circ \gamma_0$ concludes the proof. □ □

Example 3.7. *In this example we demonstrate that the application of conformal transformations does not in general preserve the property "coordinate functions are the first eigenfunctions."*

*Let $\mathcal{S}$ be a Bolza surface and let $\Pi\colon \mathcal{S} \to \mathbb{S}^2$ be the corresponding hyperelliptic projection. By [**89**], Π is given by first eigenfunctions. Let us consider instead $\Pi_t = \gamma_t \circ \Pi$, where $\gamma_t = \dfrac{z+it}{1-itz}$, $t \in [0,1)$ is a conformal transformation moving the points of $\mathbb{S}^2$ towards the point i along the shortest geodesic (the point $-i$ does not move). We claim that for t close to 1 the first eigenvalue $\lambda_1(\mathcal{S},\Pi_t^* g_{\mathrm{can}})$ is close to 0. Informally, it can be explained in the following way. As t tends to 1 the images of the branch points of Π_t are getting closer and closer together. As a result, for large t the surface $(\Sigma,\Pi_t^* g_{\mathrm{can}})$ looks like two spheres glued together with three small cylinders. To make this argument precise, we prove the following proposition.*

Proposition 3.8. *Suppose that $\Phi\colon \Sigma \to \mathbb{S}^2$ is a holomorphic map of degree d such that the images of all the branch points lie in an open disk D_r of radius r. Then $\lambda_{d-1}(\Sigma,\Phi^* g_{\mathrm{can}}) = o(1)$ as $r \to 0$.*

Proof. Let p be a center of D_r and let π be a stereographic projection onto $\mathbb{C}$ from $-p$. Then $\pi(D_r)$ is a Euclidean ball $B_\rho(0)$ of radius $\rho = 2\tan\frac{r}{2}$ with center at 0. Note that $\rho = O(r)$ as $r \to 0$. Moreover, the variational capacity of $B_\rho(0)$ in $B_1(0)$ is $2\pi|\ln\rho|^{-1}$. Therefore, there exists a function $f_r \in H^1_0(B_1(0))$ such that $f_r \equiv 1$ on $B_\rho(0)$ and the Dirichlet energy of f_r

is $o(1)$. Let $h_r = 1 - \pi^* f_r$. Then $h_r \equiv 0$ on D_r, $h_r \equiv 1$ on a hemisphere and by conformal invariance of the Dirichlet energy

$$\int_{\mathbb{S}^2} |\nabla h_r|^2 \,\mathrm{dV}(g_{\mathrm{can}}) = o(r).$$

Outside $\Phi^{-1}(D_r)$ the map Φ is a covering map. Since $\mathbb{S}^2\backslash D_r$ is a topological disk, all its covering spaces are trivial. Therefore $\Phi^{-1}(D_r)$ coincides with d copies of $\mathbb{S}^2\backslash D_r$. Let $h_{1,r}, \dots ,h_{d,r}$ be functions h_r considered as functions on their own copy of $\mathbb{S}^2\backslash D_r$. We can extend them by zero and consider as functions on Σ. Then their support is disjoint and

$$\frac{\int_\Sigma |\nabla h_{i,r}|^2 \,\mathrm{dV}(\Phi^* g_{\mathrm{can}})}{\int_\Sigma h_{i,r}^2 \,\mathrm{dV}(\Phi^* g_{\mathrm{can}})} = \frac{\int_{\mathbb{S}^2} |\nabla h_r|^2 \,\mathrm{dV}(g_{\mathrm{can}})}{\int_{\mathbb{S}^2} h_r^2 \,\mathrm{dV}(g_{\mathrm{can}})} \leq \frac{o(1)}{2\pi} = o(1).$$

The standard argument with min-max characterization of the eigenvalues concludes the proof. □ □

We see that for t close to 1 *all branch points of* Π_t *will concentrate in a small disk around* i. *Since* $\deg \Pi_t = 2$ *the previous proposition yields that* $\lambda_1(\mathcal{S}, \Pi_t^* g_{\mathrm{can}}) \to 0$ *as* $t \to 1$. *Note that this argument works with the point* i *replaced by an arbitrary point distinct from the branch point of* Π.

Example 3.9. *In this example we use the results of Example 3.7 to show that in Theorem 1.4 for conformal classes of category 3) the metric* g *is not necessarily unique. Indeed, for the Bolza surface* $\mathcal{S}$ *one has* $\lambda_4(\mathcal{S},\Pi^* g_{\mathrm{can}}) > 2 = \lambda_3(\mathcal{S},\Pi^* g_{\mathrm{can}}) = \lambda_1(\mathcal{S},\Pi^* g_{\mathrm{can}})$. *Therefore, the continuity of eigenvalues implies that for small enough* t *one has* $\lambda_4(\mathcal{S},\Pi_t^* g_{\mathrm{can}}) > 2 = \lambda_3(\mathcal{S},\Pi_t^* g_{\mathrm{can}}) = \lambda_1(\mathcal{S},\Pi_t^* g_{\mathrm{can}})$, *i.e. the immersion* Π_t *is by first eigenfunctions. Next we show that in the family* $g_t = \Pi_t^* g_{\mathrm{can}}$ *there many non-isometric metric. Indeed, if there is an isometry* τ *such that* $\tau^* g_t = g_s$, *then* τ *is conformal. At the same time, the group of conformal automorphisms of any Riemann surface of genus* $\gamma \geqslant 2$ *is finite. Since there are infinitely many metrics in the family* g_t *we conclude that there infinitely many non-isometric metrics in that family.*

Proposition 3.10. *Let* $\Psi\colon \Sigma \to \mathbb{S}^2$ *be a conformal map. Suppose* $\Phi\colon \Sigma \to \mathbb{S}^n$ *is a brabched minimal immersion such that* $\Phi^* g_{\mathrm{can}} = \Psi^* g_{\mathrm{can}}$. *Then the image of* Φ *lies in an equatorial* 2*-sphere.*

Proof. Let $g = \Psi^* g_{\mathrm{can}} = \Phi^* g_{\mathrm{can}}$. Let II denote the second fundamental form of Φ and $R^{\mathbb{S}^n}$, $R^{\Phi(\Sigma)}$ denote the Riemann tensors of $(\mathbb{S}^n,g_{can})$ and $(\Phi(\Sigma),g)$ respectively. Then the Gauss equation reads

$$\langle R^{\mathbb{S}^n}(X,Y)Z,W\rangle = \langle R^{\Phi(\Sigma)}(X,Y)Z,W\rangle + \langle II(X,Z),II(Y,W)\rangle - \langle II(Y,Z),II(X,W)\rangle,$$

for any vector fields X,Y,Z and W on $\Phi(\Sigma)$. This implies

$$1 = K + |II(X,Y)|^2 - \langle II(X,X),II(Y,Y)\rangle,$$

at any regular point of $\Phi(\Sigma)$ and any orthonormal vectors X and Y, where K is the Gauss curvature of $\Phi(\Sigma)$. Since $g = \Psi^* g_{\mathrm{can}}$ one has $K = 1$. Moreover, $II(X,X) + II(Y,Y) = 0$, since the immersion Φ is minimal. Thus, from the last equation we get:

$$|II(X,Y)|^2 + |II(X,X)|^2 = 0.$$

Therefore, the square of the norm of the second fundamental form reads:

$$|II|_g^2 := |II(X,X)|^2 + 2|II(X,Y)|^2 + |II(Y,Y)|^2 = 2(|II(X,Y)|^2 + |II(X,X)|^2) = 0,$$

which implies that $II = 0$, i.e. $\Phi(\Sigma)$ is totally geodesic in a neighborhood of any smooth point. Thus, that neighborhood gets mapped to a piece of an equatorial 2-sphere. The conclusion follows from the following standard open-closed argument.

Fix a regular point $p \in \Sigma$. Since Φ is totally geodesic in a neighbourhood U_p of p there exists a 3-dimensional subspace E_p such that $D\Phi(TU_p) \subset E_p$. Let $\Sigma_{\mathrm{reg}} \subset \Sigma$ be the set of regular points. Define V_p to be the set of $q \in \Sigma_{\mathrm{reg}}$ such that $D\Phi(T_q\Sigma) \subset E_p$. Then V_p possesses the following properties.

Non-empty. Indeed, $U_p \subset V_p$.

Open. Indeed, let $q \in V_p$. On one hand, it means that $D\Phi(T_q\Sigma) \subset E_p$. On the other, it is always true that $D\Phi(T_q\Sigma) \subset E_q$. Since $D\Phi(T_q\Sigma)$ is 2-dimensional, it follows that $E_p = E_q$. Therefore, $U_q \subset V_p$.

Closed. Indeed, the complement to V_p has the form $\cup V_q$ for some $q \in \Sigma_{\mathrm{reg}}$. Therefore, it is open.

Finally, we remark that Σ_{reg} is Σ with finitely many points removed. Thus, it is connected. Therefore, $V_p = \Sigma_{\mathrm{reg}}$ and by continuity $\Phi(\Sigma) \subset E_p$. □ □

of Theorem 1.4. Assume that Φ_1 and Φ_2 are branched minimal immersions by first eigenfunctions. Then from Proposition 3.2 we have:

$$2\,\mathrm{vol}(\Sigma, g_1) = 2\,\mathrm{vol}(\Sigma, g_2),$$

where $g_1 := \Phi_1^* g_{\mathrm{can}}$ and $g_2 := \Phi_2^* g_{\mathrm{can}}$. However, if neither of the images of Φ_1 and Φ_2 are equatorial 2-spheres then by Proposition 3.3 this is only possible if the metrics g_1 and g_2 are homothetic. Since their volumes coincide, they are equal.

Suppose that the image of the map Φ_1 lies in a 2-sphere and the image of the map Φ_2 does not. Let g_1 and g_2 be the corresponding induced metrics. First, note that $\lambda_1(g_1) = \lambda_1(g_2) = 2$. Second, by Proposition 3.2 one has $\bar\lambda_1(\Sigma, g_1) = \bar\lambda_1(\Sigma, g_2)$. Then by Proposition 3.3 applied to Φ_2 we conclude that g_1 is homothetic to g_2. Moreover, they have the same first eigenvalue, therefore, $g_1 = g_2$. The conclusion follows from Proposition 3.10. □ □

The aim of the following proposition is to show that there is no conformal class which falls into category 3) of Theorem 1.4 when the surface is a 2-torus.

Proposition 3.11. *Let Σ be a 2-torus and $\Phi : \Sigma \to \mathbb{S}^n$ be a non-constant branched minimal immersion by first eigenfunctions. Then the image of Φ cannot be an equatorial 2-sphere.*

Proof. This proposition is stated as obvious in Montiel and Ros [**79**, Corollary 8(b)]. However, we were unable to come up with an obvious explanation of this fact. Instead, we provide a proof below. Suppose that there exists a branched minimal map $\Phi\colon \mathbb{T}^2 \to \mathbb{S}^2$ by first eigenfunctions of the metric g. After possibly taking a conjugate, we may assume that Φ is holomorphic, i.e. is given by a meromorphic function f.

Claim 1. $\deg f = 2$.

By inequality (1.1), $\deg f \le 2$. At the same time, any meromorphic function of degree one is invertible which is impossible for f since $\mathbb{T}^2 \not\approx \mathbb{S}^2$.

Claim 2. For any two meromorphic functions f,h of degree 2 there exists a holomorphic automorphism γ of $\mathbb{S}^2$ such that $f = \gamma \circ h$.

This immediately follows from Proposition 3.5.

Claim 3. For any point $p \in \mathbb{T}^2$ there exists a meromorphic function f_p of degree 2 such that its only pole has degree 2 and is located at p.

Let Λ be a full rank lattice in $\mathbb{C}$ and suppose that g is conformal to the flat metric on the torus $\mathbb{C}/\Lambda$. Then we may take f_p to be $\wp(x-p)$, where $\wp$ is the Weierstrass elliptic function corresponding to Λ (for a definition of the Weierstrass elliptic function, see [**20**, Section 6.2]).

Let $p \neq q$ and let $\gamma(z) = \frac{az+b}{cz+d}$ be such that $f_p = \gamma \circ f_q$. Then

$$f_p(cf_q + d) = af_q + b \tag{3.10}$$

Claim 4. The divisor of $h = cf_q + d$ is $2p - 2q$, i.e. the only zero of h is p, its order is 2; and the only pole of h is q, its order is 2.

The function f_p has a pole of order 2 at p but the left hand side of (3.10) is finite at p. Therefore, h has a zero of order at least 2 at p. At the same time, $\deg h \le 2$, so p is the unique zero and is of order exactly 2. Similarly, h^{-1} has a unique zero of order 2 at q.

By Abel's Theorem (see [**49**, Section 5.9]), there exists h such that $(h) = 2p - 2q$ iff $2p - 2q = 0$ as points in $\mathbb{C}/\Lambda$. We arrive at a contradiction since p and q were chosen arbitrary. □ □

4. Application to the 2-torus and the Klein bottle

4.1. Conformal degeneration on the 2-torus and maximal metrics.

It is well-known that any metric on the 2-torus is conformally equivalent to a flat one obtained from the Euclidean metric on $\mathbb{C}$ under factorization by some lattice $\Gamma \subset \mathbb{C}$ generated by 1 and $a+ib \in \mathcal{M}$, where

$$\mathcal{M} := \{a+ib \in \mathbb{C} | 0 \leq a \leq 1/2, a^2+b^2 \geq 1, b > 0\}.$$

Thus, conformal classes are encoded by points of $\mathcal{M}$ (*the moduli space of flat tori*).

We point out the following action of a subgroup of the group of conformal diffeomorphisms isomorphic to $\mathbb{S}^1$. Let $\mathbb{C}/\Gamma$ where Γ is generated by 1 and $a+ib \in \mathcal{M}$. For $\theta \in \mathbb{R}$ we have an action on $\mathbb{C}$ via translation: $s_\theta(x+iy) = x+\theta+iy$. This $\mathbb{R}$-action on $\mathbb{C}$ induces an $\mathbb{S}^1$-action on $\mathbb{C}/\Gamma$ that has no fixed points. A metric in the conformal class corresponding to $a+ib \in \mathcal{M}$ is given by $f(x+iy)(dx^2+dy^2)$ where $f(z)$ is a smooth positive function that is invariant under the action of Γ. Since s_θ is a translation we have: $s_\theta^*(f(x+iy)(dx^2+dy^2)) = f(x+\theta+iy)(dx^2+dy^2)$. Thus, s_θ acts by conformal diffeomorphisms. We recall the following result concerning maximization of λ_1 and conformal degeneration.

Theorem 4.1 ([**37**])**.** *Let (g_n) be a sequence of metrics of area one on the 2-torus such that the corresponding sequence $(a_n+ib_n) \in \mathcal{M}$ satisfies $\lim_{n\to\infty} b_n = \infty$, then*

$$\lim_{n\to\infty} \lambda_1(g_n) \leq 8\pi.$$

4.2. Conformal degeneration on the Klein bottle and maximal metrics.

We define the Klein bottle as the quotient of $\mathbb{C}$ under the action of the group G_b, generated by the following elements:

$$t_b(x+iy) = x+i(y+b); \quad \tau(x+iy) = x+\pi-iy.$$

As a consequence of the Uniformization Theorem, any metric on the Klein bottle is conformal to a flat metric on $K_b := \mathbb{C}/G_b$ for some $b > 0$. Thus, the moduli space of conformal classes of metrics on the Klein bottle is encoded by the positive real numbers. Similar to the case of the 2-torus there is a group of conformal diffeomorphisms isomorphic to $\mathbb{S}^1$. Indeed, translations of the form $x+iy \mapsto x+\theta+iy$ induce an action of $\mathbb{R}/\pi\mathbb{Z}$ on K_b without fixed points. Just as above this action induces an action by conformal diffeomorphisms. We recall the following result:

Theorem 4.2 ([**37**])**.** *Let $(g_n) \subset \mathcal{R}(\mathbb{KL})$ be a sequence of metrics of area one on the Klein bottle.*

- *If $\lim_{n\to\infty} b_n = 0$, then $\limsup_{n\to\infty} \lambda_1(g_n) \leq 8\pi$.*
- *If $\lim_{n\to\infty} b_n = \infty$, then $\limsup_{n\to\infty} \lambda_1(g_n) \leq 12\pi$.*

Roughly speaking, Theorems 4.1 and 4.2 prove that the maximal metrics for the functional $\bar{\lambda}_1$ on the 2-torus and the Klein bottle must be in a conformal class which corresponds to a fundamental domain which cannot be too “long and skinny."

4.3. Continuity results

One of the classical distances considered on the moduli space of complex structures is the *Teichmüller distance.* Naturally, this distance induces a distance d_T on the space of conformal classes. In this section we show that the conformal eigenvalues

$$\Lambda_k(\Sigma,[g]) := \sup_{g' \in [g]} \lambda_k(g') \operatorname{vol}(\Sigma, g')$$

are continuous on the space of conformal classes. This fact should be well-known but we were not able to find a reference.

Here we follow [**29**]. First, we define the Teichmüller distance for orientable surfaces. We define a notion of *dilatation.* Let $f\colon \Sigma_1 \to \Sigma_2$ be an orientation-preserving homeomorphism between two Riemann surfaces which is a diffeomorphism outside a finite set of points. The function $k_f(p)$ of f at p is defined in local coordinates as $k_f(p) = \frac{|f_z|+|f_{\bar z}|}{|f_z|-|f_{\bar z}|}$. It is defined only at points where f is smooth and does not depend on the choice of coordinates. One defines the dilatation of f by the formula $K_f = ||k_f||_\infty$. If $K_f < \infty$ then one says that f is K_f-quasiconformal.

For any holomorphic quadratic differential q_1 on a Riemann surface Σ_1 its absolute value $|q_1|$ defines a flat metric with conical singularities at zeroes of q_1 compatible with the complex structure. For any non-singular point $p_0 \in \Sigma_1$ one can define *natural coordinates* $\eta = \int_{p_0}^p \sqrt{q_1}$ such that $q_1 = d\eta^2$ and $|q_1| = |d\eta|^2$, i.e. the metric is Euclidean in these coordinates. Natural coordinates are defined up to a sign and a translation.

A homeomorphism $f\colon \Sigma_1 \to \Sigma_2$ between Riemann surfaces is called a *Teichmüller mapping* if there exist holomorphic differentials q_1 on Σ_1 and q_2 on Σ_2 and a real number $K > 1$ such that

(i) f maps zeroes of q_1 to zeroes of q_2;

(ii) If p is not a zero of q_1 then with respect to a set of natural coordinates for q_1 and q_2 centered at p and $f(p)$ respectively, the mapping f can be written as

$$f(z) = \frac{1}{2}\left(\frac{K+1}{\sqrt{K}}z + \frac{K-1}{\sqrt{K}}\bar z\right),$$

or, equivalently,

$$f(x+iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y$$

In particular, a Teichmüller map has dilatation K and is smooth outside of zeroes of q_1. Moreover, in natural coordinates $|q_1| = dx^2 + dy^2$ and $f^*|q_2| = Kdx^2 + \frac{1}{K}dy^2$.

Theorem 4.3 (Teichmüller's Theorem). *Given an orientation preserving homeomorphism $f\colon \Sigma_1 \to \Sigma_2$ between non-isomorphic Riemann surfaces there exists a Teichmüller mapping homotopic to f. It is unique provided $\chi(\Sigma) < 0$. If $\chi(\Sigma) = 0$ then a Teichmüller mapping is* *affine and is unique up to a translation, therefore the dilatation is independent of the choice of the mapping.*

Definition 4.1. *Let Σ be an orientable surface. Consider two different complex structures on Σ making it into Riemann surfaces Σ_1 and Σ_2. Then one defines the* Teichmüller distance *between Σ_1 and Σ_2 as follows,*

$$d_T(\Sigma_1,\Sigma_2) = \frac{1}{2}\inf_f \log(K_f),$$

where f ranges over all Teichmüller mappings $f\colon \Sigma_1 \to \Sigma_2$.

Remark 4.2. *The fact that d_T is indeed a distance function is not obvious and relies on proper discontinuity of the action of the mapping class group on the Teichmüller space.*

Teichmüller distance d_T on the moduli space of complex structures induces a distance function on the moduli space of conformal classes. Indeed, let $[g_1]$ and $[g_2]$ be two conformal classes. Choose complex structures Σ_i compatible with $[g_i]$ inducing the same orientation on Σ. Then one sets $d_T([g_1],[g_2]) = d_T(\Sigma_1,\Sigma_2)$.

Up until now we considered orientable surfaces Σ. Let us now address the case of non-orientable Σ. Denote by $\pi\colon \hat\Sigma \to \Sigma$ an orientable double cover and by σ a corresponding involution exchanging the leaves of π. Let $[g_1]$ and $[g_2]$ be two conformal classes on Σ. Choose two complex structures $\hat\Sigma_1$ and $\hat\Sigma_2$ on $\hat\Sigma$ compatible with $[\pi^* g_1]$ and $[\pi^* g_2]$. Then one defines

$$d_T([g_1],[g_2]) = \frac{1}{2}\inf_f \log(K_f),$$

where f ranges over all Teichmüller mappings $f\colon \hat\Sigma_1 \to \hat\Sigma_2$ commuting with σ.

Remark 4.3. *This definition is implicitly making use of the equivariant version of Teichmüller's Theorem. If in Theorem 4.3 $f\circ\sigma$ is homotopic to $\sigma\circ f$ then the Teichmüller mapping can be chosen σ-equivariant. Indeed, suppose that h is the Teichmüller mapping for f. Then $h\circ\sigma$ is the Teichmüller map for $f\circ\sigma$. Similarly, $\sigma\circ h$ is the Teichmüller map for $\sigma\circ f$. If $\sigma\circ f$ is homotopic to $f\circ\sigma$, then $h\circ\sigma$ must be homotopic to $\sigma\circ h$, and by the uniqueness part of Teichmüller's Theorem we obtain $\sigma\circ h = h\circ\sigma$.*

Proposition 4.4. *The conformal eigenvalues $\Lambda_k(\Sigma,[g])$ are continuous in the distance d_T.*

Proof. We follow the notation of [**63**]. Namely, given a conformal class c of metrics and a measure μ on Σ we define the Rayleigh quotient

$$R_c(u,\mu) = \frac{\int_\Sigma |\nabla u|_g^2 \, \mathrm{dV}(g)}{\int_\Sigma u^2 \, d\mu}$$

and the eigenvalues $\lambda_k(c,\mu)$ as critical values of the Rayleigh quotient. For a comprehensive study of eigenvalues in this context, including a proof of the existence of eigenfunctions, see [**63**].

We start with the case of an orientable Σ. Let $[g_1]$ and $[g_2]$ be two conformal classes and let Σ_1 and Σ_2 be the corresponding Riemann surfaces. Denote by $f\colon \Sigma_1 \to \Sigma_2$ any Teichmüller map, let q_1 and q_2 be the corresponding quadratic differentials and suppose that S_1 and S_2 are their zeroes respectively. By property (ii) at any point of $\Sigma_1\backslash S_1$ one has

$$\frac{1}{K}f^*|q_2| \le |q_1| \le K f^*|q_2|. \tag{4.1}$$

At this point we use the conformal invariance of the Dirichlet energy. Let $K_i \subset K_{i+1}$ be a compact exhaustion of $\Sigma_2\backslash S_2$. Similarly, $f^{-1}(K_i)$ form a compact exhaustion of $\Sigma_1\backslash S_1$. Then for any $u \in H^1(\Sigma_2)$ one has

$$\int_{\Sigma_1\backslash f^{-1}(K_i)} |\nabla(f^*u)|^2_{f^*|q_2|}\,\mathrm{dV}(f^*|q_2|) = \int_{\Sigma_2\backslash K_i} |\nabla u|^2_{|q_2|}\,\mathrm{dV}(|q_2|).$$

Combining this with inequality (4.1), one obtains

$$\begin{aligned}\frac{1}{K}\int_{\Sigma_1\backslash f^{-1}(K_i)} |\nabla(f^*u)|^2_{f^*|q_1|}\,\mathrm{dV}(f^*|q_1|) \le &\int_{\Sigma_2\backslash K_i} |\nabla u|^2_{|q_2|}\,\mathrm{dV}(|q_2|)\\ \le &K\int_{\Sigma_1\backslash f^{-1}(K_i)} |\nabla(f^*u)|^2_{f^*|q_1|}\,\mathrm{dV}(f^*|q_1|)\end{aligned}$$

Passing to the limit $i \to \infty$ and using conformal invariance of the Dirichlet energy leads to

$$\frac{1}{K}\int_{\Sigma_1} |\nabla(f^*u)|^2_{g_1}\,\mathrm{dV}(g_1) \le \int_{\Sigma_2} |\nabla u|^2_{g_2}\,\mathrm{dV}(g_2) \le K\int_{\Sigma_1} |\nabla(f^*u)|^2_{g_1}\,\mathrm{dV}(g_1). \tag{4.2}$$

Let $h_2 \in [g_2]$ be a smooth metric on Σ_2. Then $\mu = (f^{-1})_* v_{h_2}$ defines a measure on Σ_1 such that

$$\int_{\Sigma_1} f^*u\,d\mu = \int_{\Sigma_2} u\,\mathrm{dV}(h_2).$$

In particular, $\mathrm{vol}(\Sigma_1,\mu) = \mathrm{vol}(\Sigma_2, h_2)$.

Putting these bounds together, we obtain that

$$\frac{1}{K}R_{[g_1]}(f^*u,\mu) \le R_{[h_2]}(u) \le K R_{[g_1]}(f^*u,\mu).$$

However, since the Teichmüller mapping f is not smooth at zeroes of q_1, the measure μ is not a volume measure of a smooth Riemannian metric. In the last step of this proof we show that there exists a sequence of metrics $\rho_n \in [g_1]$ such that $\lambda_1(\rho_n) \to \lambda_1([g_1],\mu)$. In order to do that we first obtain a local expression for μ close to the singular points.

Let s be a zero of q_1 and let z_i, $i = 1,2$ be local complex coordinates in the neighborhood of s and $f(s)$ respectively such that $q_i = z_i^k dz_i^2$. In cones with vertices at s and $f(s)$ respectively one can introduce the coordinates $\zeta_i = \frac{k+2}{2}\int_0^{z_i}\sqrt{q_i} = z_i^{\frac{k+2}{2}}$. Then in coordinates ζ_i the mapping f is linear, i.e. in appropriately chosen cones the mapping f in z_i-coordinates takes form

$$f(z_1) = \left(\tilde f(z_1^{\frac{k+2}{2}})\right)^{\frac{2}{k+2}}, \tag{4.3}$$

where $\tilde{f}$ is linear and the branch of the root function is chosen so that in the coordinate cone $z_i = \zeta_i^{\frac{2}{k+2}}$.

Now suppose that $\mathrm{dV}(h_2) = \alpha(z) dz_2 d\bar{z}_2$. Then using (4.3) one obtains

$$\begin{aligned} d\mu =& \alpha(f(z_1)) df(z_1) d\overline{f(z_1)} \\ =& \alpha(f(z_1)) \left| z_1^{-\frac{k+2}{2}} \tilde{f}(z_1^{\frac{k+2}{2}}) \right|^{-\frac{k}{k+2}} \left(|\tilde{f}_z(z_1^{\frac{k+2}{2}})|^2 - |\tilde{f}_{\bar{z}}(z_1^{\frac{k+2}{2}})|^2 \right) dz_1 d\bar{z}_1. \end{aligned}$$

As $\tilde{f}$ is linear, we conclude that $d\mu = \beta dz_1 d\bar{z}_1$ where $\beta \in L^\infty(\Sigma)$. At this point an appropriate approximation can be constructed using the following lemma and a standard regularization procedure.

Lemma 4.5. *Let g be a Riemannian metric on a surface Σ. Suppose that $\{\rho_\varepsilon\} \subset L^\infty(\Sigma)$ is an equibounded sequence such that $\rho_\varepsilon \to \rho$ as $\varepsilon \to 0$ $\mathrm{dV}(g)$-a.e. Then one has for every $k > 0$*

$$\lambda_k([g],\rho_\varepsilon \mathrm{dV}(g)) \to \lambda_k([g],\rho \mathrm{dV}(g)).$$

Therefore, taking the supremum over all h_2 yields

$$\Lambda_k(\Sigma,[g_2]) \leq K \Lambda_k(\Sigma,[g_1]).$$

Switching the role of Σ_1 and Σ_2 and considering f^{-1} instead of f in the previous argument completes the proof in the orientable case.

The proof in the non-orientable case is easily reduced to the orientable case using the following construction. For any metric g on Σ the metric $\pi^* g$ on $\hat{\Sigma}$ is σ-invariant. Thus, its eigenvalues are split into σ-even and σ-odd. Moreover, σ-even eigenvalues coincide with eigenvalues of (Σ,g). Since the Teichmüller map in this case preserves σ-even functions, one can repeat the proof of the orientable case, restricting oneself to even eigenvalues. This completes the proof, modulo the proof of the lemma. □ □

of Lemma 4.5. The proof of this lemma follows the proof of a similar statement for Steklov eigenvalues found in Lemma 3.1 of [**62**]. For completeness, we provide the proof. First, we observe there is a constant $C > 0$ that does not depend on ϵ such that $\lambda_k([g], \rho_\epsilon \mathrm{dV}(g)) \leq Ck$. This follows from the Theorem A_k on the top of page 7 of [**63**]. Moreover, by Proposition 1.1 of [**63**] we also have

$$\limsup \lambda_k([g], \rho_\epsilon \mathrm{dV}(g)) \leq \lambda_k([g], \rho \mathrm{dV}(g)).$$

Thus, it suffices to prove that $\lambda_k([g], \rho \mathrm{dV}(g)) \leq \liminf \lambda_k([g], \rho_\epsilon \mathrm{dV}(g))$. Let u_ϵ be an eigenfunction corresponding to $\lambda_k([g], \rho_\epsilon \mathrm{dV}(g))$ normalized so that $\|u_\epsilon\|_{L^2(\rho_\epsilon \mathrm{dV}(g))} = 1$. We will show that the $L^2(\mathrm{dV}(g))$ and $H^1(\Sigma,\mathrm{dV}(g))$-norms of the u_ϵ are bounded uniformly in ϵ, for $\epsilon > 0$ sufficiently small. We recall the following proposition:

Proposition 4.6. *([1] Lemma 8.3.1) Let (M,g) be a Riemannian manifold. Then there exists a constant $C > 0$ such that for all $L \in H^{-1}(M)$ with $L(1) = 1$ one has*

$$\|u - L(u)\|_{L^2(M)} \leq C\|L\|_{H^{-1}(M)} \left(\int_M |\nabla u|_g^2 \,\mathrm{dV}_g\right)^{1/2}$$

for all $u \in H^1(M)$.

We will apply Proposition 4.6 with $L_\epsilon(u) = \int_\Sigma u\rho_\epsilon \,\mathrm{dV}(g)$. First, we compute the norm of L_ϵ. We have:

$$\left|\int_\Sigma u\rho_\epsilon \mathrm{dV}(g)\right| \leq C\int_\Sigma |u|\mathrm{dV}(g) \leq C\|u\|_{L^2(\mathrm{dV}(g))} \leq C\|u\|_{H^1(\Sigma,g)},$$

where we used in order the uniform boundedness of ρ_ϵ, the Cauchy-Schwarz inequality, and the compact embedding of $H^1(\Sigma,g)$ into $L^2(\mathrm{dV}(g))$. Thus, the family of operators L_ϵ are uniformly bounded in $H^{-1}(\Sigma)$. Applying Proposition 4.6 to L_ϵ and u_ϵ yields:

$$\|u_\epsilon\|_{L^2(\mathrm{dV}(g))} \leq C\left(\int_\Sigma |\nabla u_\epsilon|^2 \mathrm{dV}(g)\right)^{1/2} = C\sqrt{\lambda_k([g],\rho_\epsilon \mathrm{dV}(g))},$$

since $L_\epsilon(u_\epsilon) = 0$. Thus, we see that u_ϵ are uniformly bounded in $H^1(\Sigma, \mathrm{dV}(g))$ and $L^2(\mathrm{dV}(g))$.

We will now show that the family u_ϵ is uniformly bounded with respect to ϵ. Indeed, each u_ϵ satisfies $\Delta_g u_\epsilon = \lambda_k([g],\rho_\epsilon \mathrm{dV}(g))\rho_\epsilon u_\epsilon$ in a weak sense. Since Δ_g is a second order elliptic differential operator by the Sobolev Embedding Theorem and elliptic regularity we have:

$$\begin{aligned}\|u_\epsilon\|_\infty \leq C\|u_\epsilon\|_{H^2(\mathrm{dV}(g))} \leq & C(\|u_\epsilon\|_{L^2(\mathrm{dV}(g))} + \|\lambda_k([g], \rho_\epsilon \mathrm{dV}(g))\rho_\epsilon u_\epsilon\|_{L^2(\mathrm{dV(g)})}) \\ \leq & C(1 + \lambda_k([g], \rho_\epsilon \mathrm{dV}(g))),\end{aligned}$$

where the last inequality comes from the fact that the L^2-norms of the u_ϵ and the L^∞-norms of ρ_ϵ are uniformly bounded. The claim follows since the eigenvalues are uniformly bounded.

Now we show that if u_ϵ and v_ϵ are $\rho_\epsilon \mathrm{dV}(g)$-orthogonal eigenfunctions then:

$$\int_\Sigma u_\epsilon^2 \rho \,\mathrm{dV}(g) \to 1 \quad \text{and} \quad \int_\Sigma u_\epsilon v_\epsilon \rho \,\mathrm{dV}(g) \to 0.$$

Since $\|u_\epsilon\|_{L^2(\rho_\epsilon \mathrm{dV}(g))} = 1$ we have:

$$\left|\int_\Sigma u_\epsilon^2 \rho \,\mathrm{dV}(g) - 1\right| \leq \int_\Sigma |u_\epsilon|^2 |\rho - \rho_\epsilon| \,\mathrm{dV}(g).$$

Since the u_ϵ are uniformly bounded, the first claim follows. Since $\int_\Sigma u_\epsilon v_\epsilon \rho_\epsilon \,\mathrm{dV}(g) = 0$ a similar argument shows that $\int_\Sigma u_\epsilon v_\epsilon \rho \,\mathrm{dV}(g) \to 0$.

Finally, let $E_{k+1}(\epsilon)$ be a direct sum of the first k eigenspaces for $([g], \rho_\epsilon \mathrm{dV}(g))$ with ρ_ϵ-orthonormal basis given by $\{u_\epsilon^i\}_{i=1}^{k+1}$. Any function in $E_{k+1}(\epsilon)$ can be written as $\sum_{i=1}^{k+1} c_i u_\epsilon^i$.

Plugging this into the Rayleigh quotient yields:

$$
\begin{aligned}
\lambda_k([g],\rho \mathrm{dV}(g)) \leq & \frac{\int_\Sigma |\sum_i c_i \nabla u_\epsilon^i|^2 \,\mathrm{dV}(g)}{\int_\Sigma (\sum_i c_i u_\epsilon^i)^2 \rho \,\mathrm{dV}(g)} \\
= & C_\epsilon \frac{\sum_i c_i^2 \int_\Sigma |\nabla u_\epsilon^i|^2 \,\mathrm{dV}(g)}{\sum_i c_i^2 \int_\Sigma (u_\epsilon^i)^2 \rho \,\mathrm{dV}(g)} \\
\leq & C_\epsilon \max_i \frac{\int_\Sigma |\nabla u_\epsilon^i|^2 \,\mathrm{dV}(g)}{\int_\Sigma (u_\epsilon^i)^2 \rho \,\mathrm{dV}(g)},
\end{aligned}
$$

where

$$
C_\epsilon = \frac{\sum_i c_i^2 \int_\Sigma (u_\epsilon^i)^2 \rho \,\mathrm{dV}(g)}{\sum_i c_i^2 \int_\Sigma (u_\epsilon^i)^2 \rho \,\mathrm{dV}(g) + \sum_{i<j} 2 c_i c_j \int_\Sigma u_\epsilon^i u_\epsilon^j \rho \,\mathrm{dV}(g)}
$$

and in the last inequality we made use of the inequality $\frac{x_1+x_2}{y_1+y_2} \leq \max\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right)$, for positive real numbers x_1,x_2, y_1 and y_2. By our previous observations $C_\epsilon \to 1$ while the numerator of the last term in the inequality is $\lambda_k([g], \rho_\epsilon \mathrm{dV}(g))$ and the denominator goes to one. Thus, we have $\lambda_k([g],\rho \mathrm{dV}(g)) \leq \liminf \lambda_k([g], \rho_\epsilon \mathrm{dV}(g))$, which completes the proof. □ □

4.4. Proof of Theorem 1.5

The proof contains two short steps. First, we prove that the values $\Lambda_1(\mathbb{T}^2)$ and $\Lambda_1(\mathbb{KL})$ are achieved by metrics smooth away from finitely many conical singularities. Second, we apply Theorem 1.4 to prove that these metrics cannot have conical points, i.e. they must be smooth everywhere.

Proof. **Step 1.** As we discussed in Sections 4.1 and 4.2 the space of conformal classes $\mathbb{T}^2$ and $\mathbb{KL}$ can be identified with subsets of $\mathbb{R}$ and $\mathbb{C}$ respectively. Moreover, the induced topology coincides with the topology generated by Tecihmüller distance, see [**29**].

Let Σ denote either $\mathbb{T}^2$ or $\mathbb{KL}$ and $\{g_n\}$ be a sequence of metrics on Σ such that $\lim \bar\lambda_1(\Sigma,g_n) \to \Lambda_1(\Sigma)$. From [**80**] and [**47**] we know that:

$$\Lambda_1(\mathbb{T}^2) > 8\pi$$

and

$$\Lambda_1(\mathbb{KL}) \geq 12\pi E(2\sqrt{2}/3) > 12\pi.$$

Therefore, by Theorems 4.1, 4.2 the conformal classes $[g_n]$ belong to a compact subset of the space of conformal classes. Thus, the sequence $\{[g_n]\}$ has a limit point $[g]$. By Proposition 4.4 one has $\Lambda_1(\Sigma,[g]) = \Lambda_1(\Sigma)$. It was proved by Petrides [**96**] that for any conformal class $[h]$ there exists a metric $\widetilde{h} \in [h]$, smooth except maybe at a finite number of singular points corresponding to conical singularities, such that $\Lambda_1(\Sigma,[h]) = \bar\lambda_1(\Sigma, \widetilde{h})$. We conclude that there exists a metric $\widetilde{g} \in [g]$ such that $\Lambda_1(\Sigma) = \bar\lambda_1(\Sigma,\widetilde{g})$. Moreover, by Theorem

1.3 and Remark 1.4 $\widetilde{g}$ is induced from a (possibly branched) minimal immersion by first eigenfunctions Φ of Σ into a round sphere of dimension at least three.

Step 2. Suppose that $\widetilde{g}$ has a conical point. From Theorem 1.3 it follows that this metric is induced from a branched minimal isometric immersion into a round sphere. As it was observed in sections 4.1 and 4.2, on Σ there exist natural $\mathbb{S}^1$-actions s_θ by conformal diffeomorphisms without fixed points. Then the mapping $\Phi \circ s_\theta$ is again a branched minimal immersion. The metric induced by this immersion is $s_\theta^* \widetilde{g}$. Therefore, since the components of Φ are the first eigenfunctions of $(\Sigma,\widetilde{g})$, then the components of $s_\theta^* \Phi = \Phi \circ s_\theta$ are the first eigenfunctions of $(\Sigma, s_\theta^* \widetilde{g})$. By Theorem 1.4 the metrics $s_\theta^* \widetilde{g}$ and $\widetilde{g}$ must be equal. Thus, a $\bar{\lambda}_1$-maximal metric must be a metric of revolution. Under this $\mathbb{S}^1$-action the conical point forms a 1-dimensional singular set, which contradicts Step 1 (the set of conical points of Λ_1-maximal metric is at most finite). This completes the proof of the theorem. □ □

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Second Chapter.

On the Friedlander-Nadirashvili invariants of surfaces

by

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Abstract. Let M be a closed smooth manifold. In 1999, L. Friedlander and N. Nadirashvili introduced a new differential invariant $I_1(M)$ using the first normalized nonzero eigenvalue of the Lalpace-Beltrami operator Δ_g of a Riemannian metric g. They defined it taking the supremum of this quantity over all Riemannian metrics in each conformal class, and then taking the infimum over all conformal classes. By analogy we use k-th eigenvalues of Δ_g to define the invariants $I_k(M)$ indexed by positive integers k. In the present paper the values of these invariants on surfaces are investigated. We show that $I_k(M) = I_k(\mathbb{S}^2)$ unless M is a non-orientable surface of even genus. For orientable surfaces and $k = 1$ this was earlier shown by R. Petrides. In fact L. Friedlander and N. Nadirashvili suggested that $I_1(M) = I_1(\mathbb{S}^2)$ for any surface M different from $\mathbb{RP}^2$. We show that, surprisingly enough, this is not true for non-orientable surfaces of even genus, for such surfaces one has $I_k(M) > I_k(\mathbb{S}^2)$. We also discuss the connection between the Friedlander-Nadirashvili invariants and the theory of cobordisms, and conjecture that $I_k(M)$ is a cobordism invariant.
Keywords: Conformal spectrum, Friedlander-Nadirshvili invariants, moduli space of conformal classes, cobordisms

1. Introduction

1.1. Preliminaries

Let (M,g) be a closed n-dimensional Riemannian manifold. Consider *the Laplace-Beltrami operator* $\Delta = -div_g \circ grad_g$. It is an elliptic self-adjoint operator of second order. Its spectrum is a discrete collection of non-negative eigenvalues with finite multiplicities,

$$0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g) \leqslant ... \nearrow +\infty.$$

We are interested in studying the extremal properties of $\lambda_k(g)$. To this end we consider $\lambda_k(g)$ as a functional on the space $\mathcal{R}(M)$ of Riemannian metrics on M,

$$\lambda_k \colon \mathcal{R}(M) \to \mathbb{R}_+,$$
$$g \mapsto \lambda_k(g).$$

However, it turns out that for any positive constant $t > 0$ one has

$$\lambda_k(tg) = \frac{\lambda_k(g)}{t},$$

which is not convenient for our purposes. Instead, we consider *normalized eigenvalues* defined by

$$\overline{\lambda}_k(M,g) = \lambda_k(g) Vol(M,g)^{\frac{2}{n}},$$

where $Vol(M,g)$ stands for the volume of the Riemannian manifold (M,g).

Theorem 1.1 ([**65, 12, 43**]). *One has the following bounds*

- *If* $\dim M = 2$*, then there exists a constant* $C > 0$ *depending only on the topology of* M *such that*
$$\overline{\lambda}_k(M,g) \leqslant Ck.$$
- *If* $\dim M \geqslant 3$*, then the functional* $\overline{\lambda}_k(M,g)$ *is not bounded from above on the space* $\mathcal{R}(M)$*.*
- *In any dimension there exists a constant* $C([g]) > 0$ *depending only on the conformal class* $[g] = \{e^{2\omega}g|\ \omega \in C^\infty(M)\}$ *such that for every metric* $\tilde{g} \in [g]$ *one has*
$$\overline{\lambda}_k(M,\tilde{g}) \leqslant Ck^{\frac{2}{n}}.$$

Remark 1.2. *Theorem 1.1 holds for any compact manifold with smooth boundary if we replace* $\overline{\lambda}_k(M,g)$ *by* $\overline{\lambda}_k^N(M,g) = \lambda_k^N(g)Vol(M,g)^{\frac{2}{n}}$*, where* $\lambda_k^N(g)$ *is the* k*-th Neumann eigenvalue of the metric* g*.*

Theorem 1.1 guarantees that the following quantities are finite
$$\Lambda_k(M) = \sup_{g\in\mathcal{R}(M)} \overline{\lambda}_k(M,g),$$
if $\dim M = 2$;
$$\Lambda_k(M,[g]) = \sup_{\tilde{g}\in[g]} \overline{\lambda}_k(M,\tilde{g}),$$
in any dimension. If $\dim M = 2$, then we will often use the notation Σ instead of M.

The invariant $\Lambda_k(\Sigma)$ has been studied extensively in the last years (see, for example [**44, 70, 96, 98, 82, 56, 89, 80, 24, 73, 38**] and references therein). The invariant $\Lambda_k(M,[g])$ is less studied (see, for instance [**96, 25, 13, 58**]). Below we recall some result which are relevant to our exposition.

Theorem 1.3 ([**97, 96**])**.** *Let* (M,g) *be a closed* n*-dimensional Riemannian manifold not conformally diffeomorphic to the sphere* $(\mathbb{S}^n,g_{can})$*, where* g_{can} *is the standard round metric on* $\mathbb{S}^n$*. Then one has*
$$\Lambda_1(M,[g]) > \Lambda_1(\mathbb{S}^n,[g_{can}]).$$

Theorem 1.4 ([**13**])**.** *For every Riemannian metric* g *on a closed* n*-dimensional manifold* M *one has*
$$\Lambda_k(M,[g]) \geqslant \Lambda_k(\mathbb{S}^n,[g_{can}]) \tag{1.1}$$
and
$$\Lambda_k(M,[g])^{\frac{n}{2}} \geqslant \Lambda_{k-1}(M,[g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n,[g_{can}]). \tag{1.2}$$

1.2. Main results

In this paper we investigate the functional

$$I_k(M) = \inf_{[g]} \Lambda_k(M,[g]),$$

called *the Friedlander-Nadirashvili invariant.* It is a differential invariant depending only on the smooth structure on M.

Let us briefly describe the history of this functional. The invariant $I_1(M)$ was introduced in the paper [**34**], where Friedlander and Nadirashvili proved that for every n-dimensional closed manifold M one has

$$I_1(M) \geqslant \overline{\lambda}_1(\mathbb{S}^n, g_{\mathrm{can}}).$$

In particular, if Σ is a closed surface then

$$I_1(\Sigma) \geqslant 8\pi.$$

Inequalities (1.1) and (1.2) imply that

$$I_k(M) \geqslant \Lambda_k(\mathbb{S}^n, [g_{can}]) \tag{1.3}$$

and

$$I_k(M)^{\frac{n}{2}} \geqslant I_{k-1}(M)^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n, [g_{can}]). \tag{1.4}$$

We introduce the following notations. Let $\widetilde{\Sigma}_\gamma$ denote an orientable closed surface of genus γ and Σ_γ denote a non-orientable closed surface of genus γ. Here the genus of a non-orientable closed surface is defined to be the genus of its orientable double cover. Furthermore we set $I_k(\gamma) = I_k(\Sigma_\gamma)$ and $\widetilde{I_k}(\gamma) = I_k(\widetilde{\Sigma}_\gamma)$. In general, we use tilde for anything related to orientable surfaces and do not use it otherwise.

Let us recall known results. Since any two metrics on $\mathbb{S}^2$ or $\mathbb{RP}^2$ are conformally equivalent, one has $I_k(0) = \Lambda_k(\mathbb{RP}^2)$ and $\widetilde{I}_k(0) = \Lambda_k(\mathbb{S}^2)$. According to [**56**], $\Lambda_k(\mathbb{S}^2) = 8\pi k$. Similarly, it was proved in [**53**] that $\Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$. For historical review in research of the invariants $\Lambda_k(\mathbb{S}^2)$ and $\Lambda_k(\mathbb{RP}^2)$ see the survey [**95**].

In the paper [**34**] Nadirashvili and Friedlander suggested that $I_1(M) = 8\pi$ for any closed surface M other than the projective plane. This statement was confirmed in certain cases. In the paper [**37**] Girouard proved that $I_1(\mathbb{KL}) = I_1(\mathbb{T}^2) = I_1(\mathbb{S}^2) = 8\pi$, where $\mathbb{KL}$ is the Klein bottle (see also [**80**]). Petrides in the paper [**96**] extended the ideas of Nadirashvili and Girouard and proved that if M is a smooth compact *orientable* surface then $I_1(M) = 8\pi$ and the infimum is attained only on the sphere $\mathbb{S}^2$.

The main result of this paper is the following theorem.

Theorem 1.5. *The following statements hold.*

(i) The Friedlander-Nadirashvili invariants of orientable surfaces satisfy $\widetilde{I}_k(\gamma) = \widetilde{I}_k(0) = 8\pi k$ *for any* $\gamma \geqslant 0$*. The infimum is attained iff* $\gamma = 0$*.*

(ii) The Friedlander-Nadirahsvili invariants of non-orientable surfaces of odd genus $\gamma \geqslant 1$ *satisfy* $I_k(\gamma) = \widetilde{I}_k(0) = 8\pi k$*. The infimum is never attained.*

(iii) The Friedlander-Nadirashvili invariants of non-orientable surfaces of even genus $\gamma \geqslant 2$ *satisfy*

$$I_k(\gamma) \leqslant I_k(\gamma - 2) \tag{1.5}$$

If inequality (1.5) *is strict, then there exists a conformal class* c *such that* $I_k(\gamma) = \Lambda_k(\Sigma_\gamma,c)$*.*

Corollary 1.6. *If* $\gamma \geqslant 2$ *is even, then one has*

$$8\pi k = \widetilde{I}_k(0) < I_k(\gamma) \leqslant I_k(0) = 4\pi(2k+1).$$

In particular, for $k = 1$ *one has*

$$8\pi < I_1(\gamma) \leqslant 12\pi,$$

for all even γ*.*

Therefore, Corollary 1.6 shows that the statement "$I_1(M) = 8\pi$ unless M is a projective plane" suggested by Friedlander and Nadirashvili in [**34**] does not hold for non-orientable surfaces of even genus.

The main idea in the proof of Theorem 1.5 is to investigate the behaviour of the quantity $\Lambda_k(M,c_n)$ when the sequence of conformal classes $\{c_n\}$ escapes to infinity in the moduli space of conformal classes on M. The precise expression for the limit makes use of Deligne-Mumford compactification. It is stated in Theorem 2.11 and is proved in Section 5.

As a byproduct of our approach we obtain a result on conformal Neumann eigenvalues that could be of independent interest. Consider a smooth domain Ω in M. Then we define the following functional

$$\Lambda_k^N(\Omega,[g|_\Omega]) := \sup_{\tilde{g}\in[g|_\Omega]} \overline{\lambda}_k^N(\Omega,\tilde{g}),$$

where $\overline{\lambda}_k^N(\Omega,\tilde{g}) = \lambda_k^N(\Omega,\tilde{g})Vol(\Omega,\tilde{g})^{\frac{2}{n}}$ and $\lambda_k^N(\Omega,\tilde{g})$ is the k-th Neumann eigenvalue of the domain Ω in the metric $\tilde{g}$. In the sequel we often omit the restriction symbol and simply write $\Lambda_k^N(\Omega,[g])$.

Proposition 1.7. *Let* (M,g) *be a compact Riemannian manifold and* $\Omega \subset M$ *be a smooth domain. Then the following inequality holds,*

$$\Lambda_k(M,[g]) \geqslant \Lambda_k^N(\Omega,[g]).$$

Remark 1.8. *Similar results for analogs of the Friedlander-Nadirashvili invariants for the Steklov problem have been recently obtained by the second named author in the paper* [**75**].

1.3. Discussion

One of the questions that Corollary 1.6 leaves unanswered is the exact value of $I_k(\gamma)$ for even γ. By an analogy with Theorem 1.5, (i) and (ii), the following conjecture seems natural.

Conjecture 1.9. *For all even γ one has*

$$I_k(\gamma) = I_k(0).$$

The infimum is attained iff $\gamma = 0$.

Another natural question is: why do the quantities $I_k(\gamma)$ take different values for odd and even γ? Careful analysis of the proof suggests that the answer lies in the theory of *cobordisms*. We recall that two closed manifolds M and M' of the same dimension are called *cobordant* if there exists a manifold with boundary W such that the boundary ∂W is the disjoint union $M \sqcup M'$. Similarly, M is cobordant to 0 or null cobordant if there exists W such that $\partial W = M$. One of the basic facts of cobordism theory is that two manifolds are cobordant iff they can be obtained from one another by a sequence of surgeries, see e.g. [**76**]. In dimension 2 it implies that attaching a handle does not change the cobordism class. This makes the cobordism theory for surfaces rather straightforward. Indeed, since $\mathbb{S}^2$ and $\mathbb{KL}$ are obviously cobordant to 0, one concludes that all orientable surfaces and all non-orientable surfaces of odd genus are cobordant to 0. By the same token, all non-orientable surfaces of even genus are cobordant to $\mathbb{RP}^2$. The fact that $\mathbb{RP}^2$ is not cobordant to 0 can be shown using Stiefel-Whitney characteristic classes, see e.g. [**77**].

Assuming Conjecture 1.9, the quantity I_k is a *cobordism invariant* in dimension 2. Inequality (1.5) can be interpreted as monotinicity of I_k with respect to addition of a handle. The monotonicity then can be shown by choosing a degenerate sequence of conformal classes such that the handle collapses in the limit. It turns out that for such sequence the functional $\Lambda_k(M,c)$ is continuous, see Remark 2.12. We believe that the same phenomenon occurs in higher dimensions and propose the following extension of Conjecture 1.9.

Conjecture 1.10. *The quantities I_k are cobordism invariants, i.e. if M is cobordant to M' then $I_k(M) = I_k(M')$. In particular, if M is cobordant to 0 then $I_k(M) = I_k(\mathbb{S}^{\dim M}) = \Lambda_k(\mathbb{S}^{\dim M},[g_{can}])$.*

We remark that the cobordism theory has been used by Jammes in the paper [**48**] to study upper bounds on I_1. We plan to tackle Conjectures 1.9, 1.10 in the subsequent papers.

Notation

Let us remind the reader that $\widetilde{\Sigma}_\gamma$ denotes an orientable closed surface of genus γ and Σ_γ denotes a non-orientable closed surface of genus γ, $I_k(\gamma) = I_k(\Sigma_\gamma)$ and $\widetilde{I_k}(\gamma) = I_k(\widetilde{\Sigma}_\gamma)$. In general, we use tilde to denote anything related to orientable objects. For example, $\pi\colon \widetilde{\Sigma}_\gamma \to \Sigma_\gamma$ denotes an orientable double cover. Moreover, the notation Σ is usually used to denote a non-orientable surface and $\widetilde{\Sigma}$ is used to denote an orientable surface. If we do not want to specify orientablity of the surface, we denote it by M.

Plan of the paper.

The paper is organized in the following way. In Section 2 we provide the geometric background, including hyperbolic surfaces and the convergence on the space of hyperbolic structures on a given surface. There we state the main technical result of the paper – Theorem 2.11. In Section 3 we deduce Theorem 1.5 from Theorem 2.11 and prove Corollary 1.6. Sections 4 and 5 are devoted to proving Theorem 2.11. In Section 4 we recall necessary facts about Neumann eigenvalues and, finally, in Section 5 we complete the proof.

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2. Moduli space of conformal classes

In this section we recall necessary background on the geometry of moduli space of conformal classes on a fixed surface M. Even though the contents of this section are mostly classical, we felt inclined to include it in the paper due to the fact that the case of non-orientable surfaces is less known. In our exposition we follow the books [**8, 45**].

The starting point is the uniformization theorem that states that in any conformal class there exists a unique (up to an isometry) metric of constant Gauss curvature and fixed area. Note that the area assumption is unnecessary unless $\chi(M) = 0$ in which case we fix the volume to be equal to 1. We start with the case $\chi(M) < 0$ corresponding to hyperbolic metrics.

2.1. Orientable hyperbolic surfaces: collar theorem

We start with the definition.

Definition 2.1. *A Riemannian metric h of constant Gaussian curvature -1 is called* hyperbolic*. A Riemannian surface (M,h) endowed with a hyperbolic metric h is called* a hyperbolic surface*.*

Note that a hyperbolic surface necessarily has negative Euler characteristic. We recall one of the underlying facts of this theory: the *Collar Theorem*. Orientable case is well-known and can be found e.g. in [**8**].

Definition 2.2. *A compact Riemann surface Y of genus 0 with 3 boundary components is called* a pair of pants*.*

Theorem 2.3 (Collar theorem)**.** *Let $(\widetilde{\Sigma},h)$ be an orientable compact hyperbolic surface of genus $\gamma \geqslant 2$ and let $c_1,c_2,\ldots,c_m$ be pairwise disjoint simple closed geodesics on $(\widetilde{\Sigma},h)$. Then the following holds*

- $m \leqslant 3\gamma - 3$.
- *There exist simple closed geodesics $c_{m+1},\ldots,c_{3\gamma-3}$ which, together with $c_1,\ldots,c_m$, decompose $\widetilde{\Sigma}$ into pairs of pants.*
- *The collars*
$$\mathcal{C}(c_i) = \left\{p \in \widetilde{\Sigma} \mid dist(p,c_i) \leqslant w(c_i)\right\}$$
of widths
$$w(c_i) = \frac{\pi}{l(c_i)}\left(\pi - 2\arctan\left(\sinh\frac{l(c_i)}{2}\right)\right)$$
are pairwise disjoint for $i = 1,\ldots,3\gamma-3$.
- *Each $\mathcal{C}(c_i)$ is isometric to the cylinder $\{(t,\theta)| - w(c_i) < t < w(c_i), \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ with the Riemannian metric*
$$\left(\frac{l(c_i)}{2\pi\cos\left(\frac{l(c_i)}{2\pi}t\right)}\right)^2\left(dt^2 + d\theta^2\right).$$

The decomposition of $(\widetilde{\Sigma},h)$ into pair of pants is called *the pants decomposition.* We denote it by $\mathcal{P}$. We say that the geodesics $c_1,\ldots,c_{3\gamma-3}$ form $\mathcal{P}$.

2.2. Non-orientable hyperbolic surfaces: collar theorem

In this section we discuss the case of non-orientable surfaces. Let (Σ,h) be a non-orientable hyperbolic surface and let $\pi : \widetilde{\Sigma} \to \Sigma$ be the orientable double cover. Lifting the metric h to $\widetilde{\Sigma}$ we get an orientable hyperbolic surface $(\widetilde{\Sigma},\pi^*h)$. If τ is the involution exchanging the leaves of π, then τ is an isometry of $(\widetilde{\Sigma}, \pi^*h)$. In other words, the hyperbolic surface $(\widetilde{\Sigma}, \pi^*h)$ is τ-invariant.

Let c be a simple closed geodesic on (Σ,h). The preimage $\pi^{-1}(c)$ is either a τ-invariant simple closed geodesic $\widetilde{c}$ on $(\widetilde{\Sigma},\pi^*h)$ or a pair $\widetilde{c}_1$, $\widetilde{c}_2$ of simple closed geodesics such that $\tau(\widetilde{c}_1) = \widetilde{c}_2$. Assume $\pi^{-1}(c) = \widetilde{c}$. Then τ acts on the collar $\mathcal{C}(\widetilde{c})$ as an isometry $(t,\theta) \to (-t,\theta + \pi)$. Therefore, the π-image of the cylinder $\mathcal{C}(\widetilde{c})$ is a Möbius band $\mathcal{C}(\widetilde{c})/\tau$ around c. We refer to this Möbius band as a collar $\mathcal{C}(c)$ of c and call c a 1-*sided geodesic*. Now, assume $\pi^{-1}(c) = \widetilde{c}_1 \cup \widetilde{c}_2$. Then τ exchanges the collars $\mathcal{C}(\widetilde{c}_1)$ and $\mathcal{C}(\widetilde{c}_2)$ and their π-image is a cylinder around c. We refer to that cylinder as a collar $\mathcal{C}(c)$ of c and call c a 2-*sided geodesic*. With that we can state the collar theorem in the non-orientable case.

Theorem 2.4 (Collar theorem). *Let (Σ,h) be a compact non-orientable hyperbolic surface of genus $\gamma \geqslant 2$ and let $c_1^1,c_2^1,\dots,c_{m_1}^1,c_1^2,\dots,c_{m_2}^2$ be pairwise disjoint simple closed geodesics on (Σ,h), where c_i^1 are 1-sided geodesics and c_j^2 are 2-sided geodesics. Then the following holds*

- $m_1 + 2m_2 \leqslant 3\gamma - 3$.
- *There exist simple closed geodesics $c_{m_1+1}^1,\dots,c_{n_1}^1,c_{m_2+1}^2,\dots,c_{n_2}^2$ which, together with $c_1^1,c_2^1,\dots,c_{m_1}^1,c_1^2,\dots,c_{m_2}^2$, decompose Σ into pairs of pants. Moreover, c_i^1 are 1-sided geodesics, c_j^2 are 2-sided geodesics and $n_1 + 2n_2 = 3\gamma - 3$.*
- *The collars*

$$\mathcal{C}(c_i^\alpha) = \left\{p \in \widetilde{\Sigma} \mid dist(p,c_i^\alpha) \leqslant w(c_i^\alpha)\right\}$$

  of widths

$$w(c_i^\alpha) = \frac{\pi}{\alpha l(c_i^\alpha)}\left(\pi - 2\arctan\left(\sinh\frac{\alpha l(c_i^\alpha)}{2}\right)\right)$$

  are pairwise disjoint for $i = 1,\dots,3\gamma - 3$, $\alpha = 1,2$.
- *Each $\mathcal{C}(c_i^2)$ is isometric to the cylinder $\{(t,\theta)| - w(c_i^1) < t < w(c_i^1), \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ with the Riemannian metric*

$$\left(\frac{l(c_i^2)}{2\pi\cos\left(\frac{l(c_i^2)}{2\pi}t\right)}\right)^2\left(dt^2 + d\theta^2\right).$$

- *Each $\mathcal{C}(c_i^1)$ is isometric to the Möbius band $\{(t,\theta)| - w(c_i^2) < t < w(c_i^2), \theta \in \mathbb{R}/2\pi\mathbb{Z}\}/\sim$, where $(t,\theta) \sim (-t,\theta + \pi)$, with the Riemannian metric*

$$\left(\frac{l(c_i^1)}{\pi\cos\left(\frac{l(c_i^1)}{\pi}t\right)}\right)^2\left(dt^2 + d\theta^2\right).$$

Proof. We consider the preimages of all the geodesics on the orientable double cover $\widetilde{\Sigma}$. We then have a τ-invariant set of simple closed geodesics on $\widetilde{\Sigma}$. It is proved in the paper [**9**] that every τ-invariant set of simple closed geodesics can be complemented to the τ-invariant set of $3\gamma - 3$ simple closed geodesics. This proves (i). The rest follows from the orientable Collar theorem and the discussion above. $\square$

2.3. Convergence of hyperbolic metrics: orientable case

In this section we recall compactness properties of hyperbolic metrics. Our exposition essentially follows the book [**45**]. Let $\widetilde{\Sigma}$ be an orientable surface of genus $\gamma \geqslant 2$ and let $\{h_n\}$ be a sequence of hyperbolic metrics on $\widetilde{\Sigma}$.

Proposition 2.5 (Mumford's compactness theorem)**.** *Assume that the injectivity radii* $inj(\widetilde{\Sigma},h_n)$ *satisfy* $\limsup\limits_{n\to\infty} inj(\widetilde{\Sigma},h_n) > 0$. *Then there exists a subsequence* $\{h_{n_k}\}$, *sequence* $\{\Phi_k\}$ *of smooth automorphisms of* $\widetilde{\Sigma}$ *and a hyperbolic metric* h_∞ *on* $\widetilde{\Sigma}$ *such that the sequence of hyperbolic metrics* $\{\Phi_k^* h_{n_k}\}$ *converges in* C^∞*-topology to* h_∞.

We say that a sequence $\{h_n\}$ *degenerates* if it does not satisfy the assumptions of Mumford's compactness theorem, i.e. if $\lim\limits_{n\to\infty} inj(\widetilde{\Sigma},h_n) = 0$. We now turn to *Deligne-Mumford compactification* which allows one to associate a limiting object to a degenerating sequence of hyperbolic metrics. For the remainder of this section assume that $inj(\widetilde{\Sigma},h_n) \to 0$.

Under this assumption the thick-thin decomposition implies that for each n there exists a collection $\{c_1^n,\ldots,c_s^n\}$ of disjoint simple closed geodesics in $(\widetilde{\Sigma},h_n)$ whose lengths tend to 0. Moreover, the length of any geodesic in the complement $\widetilde{\Sigma}_n = \widetilde{\Sigma}\backslash(c_1^n\cup\ldots\cup c_s^n)$ is bounded from below by a constant independent of n. Each $(\widetilde{\Sigma}_n, h_n)$ is possibly a disconnected hyperbolic surface with geodesic boundary. Up to a choice of a subsequence all components of $\widetilde{\Sigma}_n$ have the same topological type. We denote by $\widehat{\Sigma_\infty}$ the surface having the same connected components as $\widetilde{\Sigma}_n$, but with boundary component replaced by marked points. Each sequence $\{c_i^n\}$ gives rise to a pair of marked points $\{p_i,q_i\}$ on $\widehat{\Sigma_\infty}$, $i = 1,\ldots,s$. Let us denote by Σ_∞ the punctured surface $\widehat{\Sigma_\infty}\backslash\{p_1,q_1,\ldots,p_s,q_s\}$ and by h_∞ the complete hyperbolic metric on Σ_∞ with cusps at punctures.

Proposition 2.6 (Deligne-Mumford compactification)**.** *Let* $(\widetilde{\Sigma}, h_n)$ *be a sequence of hyperbolic surfaces such that* $inj(\widetilde{\Sigma},h_n) \to 0$. *Then up to a choice of subsequence, there exists a sequence of diffeomorphisms* $\Psi_n : \Sigma_\infty \to \Sigma_n$ *such that the sequence* $\{\Psi_n^* h_n\}$ *of hyperbolic metrics converges in* C^∞_{loc}*-topology to the complete hyperbolic metric* h_∞ *on* Σ_∞. *Furthermore, there exists a metric of locally constant curvature* $\widehat{h_\infty}$ *on* $\widehat{\Sigma_\infty}$ *such that its restriction to* Σ_∞ *is conformal to* h_∞.

Remark 2.7. *We say that* $\widehat{h_\infty}$ *has* locally *constant curvature, because* $\widehat{\Sigma_\infty}$ *could be disconnected and different connected components could have different signs of Euler characteristic.*

Remark 2.8. *For the general case of hyperbolic surfaces with boundary and cusps see* [**45**, Proposition 5.1].

When the statement of Proposition 2.6 holds for the full sequence $\{h_n\}$ we say that $(\widehat{\Sigma_\infty},\widehat{h_\infty})$ is a *limiting space* of the sequence (Σ,h_n). Similarly, we say that the limit of conformal classes $[h_n]$ is the conformal class $[\widehat{h_\infty}]$ on $\widehat{\Sigma_\infty}$.

2.4. Convergence of hyperbolic metrics: non-orientable case

To the best of our knowledge, there is no straightforward argument that allows to generalize the contents of the previous section to the non-orientable case. The natural approach is to pass to the double cover to obtain a sequence of hyperbolic τ-invariant metrics and then show that the diffeomorphisms Φ_n and Ψ_n can be chosen to commute with τ. This approach is taken for example in [**102**, Section 6]. In particular, he proves that both Proposition 2.5 and 2.6 hold for non-orientable surfaces without changes. We remark that the limiting surface $\widehat{\Sigma_\infty}$ can have orientable and non-orientable connected components.

Remark 2.9. *Any conformal class on $\widehat{\Sigma_\infty}$ can be obtained as a limit of conformal classes $[h_n]$ on Σ. Indeed, consider $\widehat{\Sigma_\infty}$ and a conformal class $[g]$ on it, marked by some metric g. Removing points p_i and q_i, we then find a hyperbolic metric h in the conformal class $[g_{|\widehat{\Sigma_\infty}\setminus\cup_{i=1}^s\{p_i,q_i\}}]$ to obtain a hyperbolic surface with cusps. Take a pants decomposition of $(\widehat{\Sigma_\infty}\setminus\cup_{i=1}^s\{p_i,q_i\},h)$ and consider singular pants, i.e. pants with cusps instead of boundary. For each $\varepsilon>0$ consider a surface with boundary obtained by replacing cusps with boundary components of length ε. Gluing the boundary component corresponding to p_i with the boundary component corresponding to q_i we obtain a hyperbolic surface (Σ,h_ε). From the construction of Deligne-Mumford compactification, it follows that $(\widehat{\Sigma_\infty},[g])$ is the limiting space of (Σ,h_ε) as $\varepsilon\to 0$.*

2.5. Moduli space in non-negative Euler characteristic

Having discussed the hyperbolic surfaces that correspond to the negative Euler characteristic, we proceed to the remaining surfaces: $\mathbb{S}^2$, $\mathbb{RP}^2$, $\mathbb{T}^2$ and $\mathbb{KL}$. In case of $\mathbb{S}^2$ and $\mathbb{RP}^2$ there is a unique conformal class of metrics and as a result the moduli space of conformal classes is a single point. We give an explicit description of the moduli space for $\mathbb{T}^2$ and $\mathbb{KL}$ below.

On the torus $\mathbb{T}^2$ the moduli space of conformal classes is a subset of $\mathbb{R}^2$ given by $\{(a,b)|\, a^2+b^2\geqslant 1,\ 0\leqslant a\leqslant 1/2\}$. To each (a,b) one can associate a lattice $\Lambda_{a,b}$ in $\mathbb{R}^2$ spanned by vectors $(1,0)$ and (a,b). Then the flat metric $g_{a,b}$ of unit volume on $\mathbb{R}^2/(b^{-\frac{1}{2}}\Lambda_{a,b})$ is a canonical representative of the corresponding conformal class. Let (a_n,b_n) be a sequence of points on the moduli space. Then this sequence has an accumulation point unless $b_n\to+\infty$.

Therefore, a degenerating sequence of conformal classes corresponds to $b_n \to +\infty$. Similarly to the hyperbolic case, for the degenerating sequence (a_n, b_n) the injectivity radius $inj(\mathbb{T}^2, g_{a_n,b_n}) \to 0$ as the length of the geodesic c_n corresponding to the vector $(b_n^{-\frac{1}{2}}, 0)$ goes to zero. Moreover, c_n has a cylindrical collar of width $\frac{1}{2}\sqrt{\frac{a_n^2+b_n^2}{b_n}}$ and the limiting space is the sphere $\mathbb{S}^2$ with its unique conformal class.

On the Klein bottle the moduli space of conformal classes is the set of positive real numbers $\mathbb{R}_+$. To each $b > 0$ one can associate a group G_b of isometries of $\mathbb{R}^2$ generated by $(x,y) \mapsto (x, y + b^{\frac{1}{2}})$ and $(x,y) \mapsto (x + b^{-\frac{1}{2}}, -y)$. Then the flat metric g_b of unit volume on $\mathbb{R}^2/G_b$ is a canonical representative of the corresponding conformal class. The sequence of points $\{b_n\}$ has an accumulation point unless $b_n \to 0$ or $b_n \to +\infty$. Therefore, there are two types of degenerating sequences of conformal classes: those corresponding to $b_n \to 0$ and those corresponding to $b_n \to +\infty$. Assume $b_n \to 0$. Then the lengths of geodesics c_n corresponding to the vector $(0, b_n^{\frac{1}{2}})$ go to zero. Moreover, c_n has a cillindrical collar of width $\frac{1}{2}b_n^{-\frac{1}{2}}$, i.e. c_n is a 2-sided geodesic, and the limiting space is the sphere $\mathbb{S}^2$ with its unique conformal class. Assume $b_n \to +\infty$. Then the lengths of geodesics d_n corresponding to the vector $(b_n^{-\frac{1}{2}}, 0)$ go to zero. Moreover, d_n has a Möbius band collar of width $\frac{1}{2}b_n^{\frac{1}{2}}$, i.e. d_n is a 1-sided geodesic, and the limiting space is the projective plane $\mathbb{RP}^2$ with its unique conformal class. Either way, $inj(\mathbb{KL}, g_{b_n}) \to 0$.

2.6. Degenerating conformal classes

From now on we no longer use c to denote geodesics and reserve the letter c to denote conformal classes.

Definition 2.10. *Let M be a surface and let $\{c_n\}$ be a sequence of conformal classes on M. Let $h_n \in c_n$ be a canonical representative, i.e. h is hyperbolic if $\chi(M) < 0$ and h is flat of unit volume if $\chi(M) = 0$. We say that c_n* degenerates *if $inj(M, h_n) \to 0$. Furthermore, if $(M, h_n) \to (\widehat{M_\infty}, \widehat{h_\infty})$ in the sense of Proposition 2.6 (if $\chi(M) < 0$) or in the sense of Section 2.5 (if $\chi(M) = 0$), then we say that c_n converges to $c_\infty = [\widehat{h_\infty}]$.*

In [**11**] it is shown that if the sequence c_n does not degenerate and converges to c then one has $\Lambda_k(M, c_n) \to \Lambda_k(M, c)$. The main technical result of the present paper establishes the value of the limit of $\Lambda_k(M, c_n)$ when the sequence of conformal classes c_n degenerates.

Theorem 2.11. *Let M be a closed compact (orientable or non-orientable) surface and let $c_n \to c_\infty$ be a degenerating sequence of conformal classes. Suppose that $\widetilde{s}$ 2-sided and s 1-sided geodesics collapse, so that the surface $\widehat{M_\infty}$ has $\widetilde{m}$ orientable components $\widetilde{\Sigma}_{\widetilde{\gamma}_i}$ of genus $\widetilde{\gamma}_i$, $i = 1, \ldots, \widetilde{m}$ and m non-orientable components Σ_{γ_j} of genus γ_j, $j = 1, \ldots, m$. Then one has*

$$\lim_{n\to\infty}\Lambda_k(M,c_n) =$$
$$\max\Big(\sum_{i=1}^{\widetilde{m}}\Lambda_{\widetilde{k}_i}(\widetilde{\Sigma}_{\widetilde{\gamma}_i},c_\infty)+\sum_{i=1}^{m}\Lambda_{k_i}(\Sigma_{\gamma_i},c_\infty)+\sum_{i=1}^{\widetilde{s}}\Lambda_{\widetilde{r}_i}(\mathbb{S}^2)+\sum_{i=1}^{s}\Lambda_{r_i}(\mathbb{RP}^2)\Big), \tag{2.1}$$

where the maximum is taken over all possible combinations of indices such that

$$\sum_{i=1}^{m}k_i+\sum_{i=1}^{\widetilde{m}}\widetilde{k}_i+\sum_{i=1}^{s}r_i+\sum_{i=1}^{\widetilde{s}}\widetilde{r}_i = k.$$

Remark 2.12. *We remark that inequality* (1.2) *implies that the terms $\Lambda_{\widetilde{r}_i}(\mathbb{S}^2)=8\pi\widetilde{r}_i$ in the r.h.s of* (2.1) *can be absorbed into the other terms. This fact together with Lemma 4.8 below allows us to formulate equality* (2.1) *in a way that resembles continuity property,*

$$\lim_{n\to\infty}\Lambda_k(M,c_n)=\max_{k-s\leqslant k'\leqslant k,k'\geqslant 0}\Big(\Lambda_{k'}(\widehat{M_\infty},c_\infty)+12\pi(k-k')\Big),$$

where we have used the fact that $\Lambda_r(\mathbb{RP}^2)=4\pi(2r+1)$. As a result, the functional $\Lambda_k(M,c_n)$ is not continuous for degenerating sequences of conformal classes as long as at least a single 1*-sided geodesic collapses.*

Remark 2.13. *A result similar to Theorem 1.7 for the Steklov problem has been recently obtained in the paper* [**75**] *(see Theorem 1.2).*

The proof of Theorem 2.11 is rather technical. We postpone it until Section 5.

2.7. Topology of the limiting space

The following purely topological lemma describes the relation between the genera of M and $\widehat{M_\infty}$.

Lemma 2.14. *(i) Let $c_n\to c_\infty$ be a degenerating sequence of conformal classes on $\widetilde{\Sigma}_{\widetilde{\gamma}}$. Suppose that $\widetilde{s}$ geodesics collapse, so that the surface $\widehat{\Sigma_{\gamma,\infty}}$ has $\widetilde{m}$ components $\widetilde{\Sigma}_{\widetilde{\gamma}_i}$ of genus $\widetilde{\gamma}_i$, $i=1,\dots,\widetilde{m}$. Then one has*

$$\widetilde{\gamma}=\widetilde{s}+|\widetilde{\Gamma}|-\widetilde{m}+1 \tag{2.2}$$

where $\widetilde{\Gamma}=\{\widetilde{\gamma}_1,\dots,\widetilde{\gamma}_{\widetilde{m}}\}$, $|\widetilde{\Gamma}|=\sum_{i=1}^{m}\widetilde{\gamma}_i$.

(ii) Let $c_n\to c_\infty$ be a degenerating sequence of conformal classes on Σ_γ. Suppose that $\widetilde{s}$ 2*-sided and s* 1*-sided geodesics collapse, so that the surface $\widehat{\Sigma_{\gamma,\infty}}$ has $\widetilde{m}$ orientable components $\widetilde{\Sigma}_{\widetilde{\gamma}_i}$ of genus $\widetilde{\gamma}_i$, $i=1,\dots,\widetilde{m}$ and m non-orientable components Σ_{γ_j} of genus γ_j, $j=1,\dots,m$. Then one has*

$$\gamma=2(\widetilde{s}+|\widetilde{\Gamma}|-\widetilde{m})+s+|\Gamma|-m+1, \tag{2.3}$$

where $\widetilde{\Gamma}=\{\widetilde{\gamma}_1,\dots,\widetilde{\gamma}_{\widetilde{m}}\},\Gamma=\{\gamma_1,\dots,\gamma_m\}$, $|\widetilde{\Gamma}|=\sum_{i=1}^{\widetilde{m}}\widetilde{\gamma}_i$ and $|\Gamma|=\sum_{i=1}^{m}\gamma_i$.

Proof. (i) The surface $\widetilde\Sigma_{\widetilde\gamma}$ is obtained from components $\widetilde\Sigma_{\widetilde\gamma_i}$ by joining them with $\widetilde s$ cylinders. Recall that Mayer–Vietoris sequence implies that if $M = M_1\cup M_2$, then the Euler characteristics satisfy the following relation, $\chi(M) = \chi(M_1)+\chi(M_2)-\chi(M_1\cap M_2)$. We apply this formula to M_1 – disjoint union of $\widetilde\Sigma_{\widetilde\gamma_i}$ with $\widetilde s_i$ holes, M_2 – disjoint union of $\widetilde s$ cylinders, M is M_1 and M_2 glued by a common boundary. Since $\sum\widetilde s_i = 2\widetilde s$, one has

$$2-2\widetilde\gamma = \chi(\widetilde\Sigma_\gamma) = \sum_j(2-2\widetilde\gamma_j-\widetilde s_j) = 2\widetilde m - 2|\widetilde\Gamma| - 2\widetilde s.$$

Rearranging the terms yields (2.2).

(ii) Non-orientable case follows from the orientable case by passing to the double cover: 2-sided collapsing geodesics lift to a pair of collapsing geodesics; 1-sided collapsing geodesics lift to a single collapsing geodesic; orientable components $\widetilde\Sigma_{\widetilde\gamma_i}$ lift to two copies of itself and non-orientable components Σ_{γ_j} lift to its orientable double cover $\widetilde\Sigma_{\gamma_j}$.

□

Corollary 2.15. *In notations of Lemma 2.14(ii) assume γ is even. Then either $s\neq 0$ or γ_i is even for some i.*

Proof. By Lemma 2.14 one has

$$\gamma = 2(\widetilde s + |\widetilde\Gamma| - \widetilde m) + s + |\Gamma| - m + 1.$$

If γ_i is odd for all i, then $|\Gamma|-m$ is even. Since γ is even, this implies $s+1$ is even, i.e. $s\neq 0$. □

We conclude this section with the following observation.

Lemma 2.16. *(i) On $\widetilde\Sigma_{\widetilde\gamma}$ there exists a degenerating sequence of conformal classes $\{c_n\}$ such that the limiting space $\widehat{\Sigma_\infty}$ is a union of spheres.*

(ii) Let Σ_γ be a non-orientable surface of odd genus γ. Then there exists a degenerating sequence of conformal classes $\{c_n\}$ such that all the collapsing geodesics are 2-sided and the limiting space $\widehat{\Sigma_\infty}$ is a union of spheres.

(iii) Let Σ_γ be a non-orientable surface of even genus $\gamma\geqslant 2$. Then for any even $\gamma'<\gamma$ and any conformal class c on $\Sigma_{\gamma'}$ there exists a degenerating sequence of conformal classes $\{c_n\}$ such that all the collapsing geodesics are 2-sided and the limiting space $\widehat{\Sigma_\infty}$ is a union of spheres and a surface $\Sigma_{\gamma'}$ equipped with a conformal class c.

Proof. From the discussion in Section 2.5 this lemma is obvious in the non-negative Euler characteristic. In the remainder of the proof we focus on hyperbolic surfaces.

Consider a hyperbolic orientable surface $(\widetilde\Sigma_\gamma,h)$ of genus γ. Given a pants decomposition $\mathcal P$ of $(\widetilde\Sigma_\gamma,h)$ (see e.g. Figure 1), one can construct a new hyperbolic metric h_ε by replacing all pants in $\mathcal P$ by pants whose boundaries are scaled by ε. Sending ε to 0 gives the required sequence.

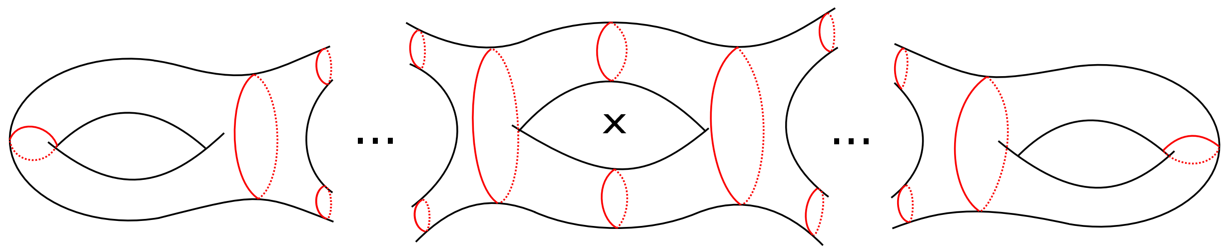

Fig. 2. Involution-invariant pants decomposition for an orientable double cover of a non-orientable surface of odd genus. The involution is given by the reflection with respect to the center point. Sending the lengths of all geodesics in the decomposition to zero provides the sequence required to prove (ii).

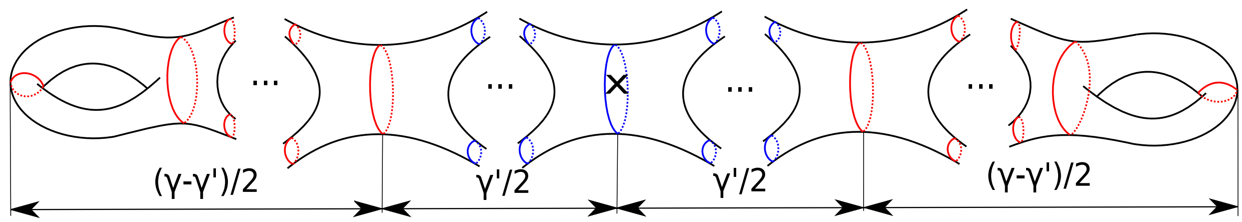

Fig. 3. Involution-invariant pants decomposition for an orientable double cover of a non-orientable surface of even genus. The involution is given by the reflection with respect to the center point. Sending lengths of all *red* geodesics in the decomposition to zero provides the sequence required to prove (iii).

To show (ii) we refer to Figure 1. It pictures a particular pants decomposition of the orientable double cover with the involution given by a reflection with respect to the center point. We see that the involution exchanges pairs of geodesics, i.e. all geodesics in the pants decomposition are 2-sided. Sending their lengths to 0 provides the required sequence.

To show (iii) we refer to Figure 2. Once again, it pictures a particular pants decomposition of the orientable double cover with the involution given by a reflection with respect to the center point. The numbers on the bottom refer to the number of handles in the marked interval. The only 1-sided geodesic is the one corresponding to the central blue geodesic, i.e. all red geodesics project onto 2-sided geodesics. Sending the lengths of all *red* geodesics in the the decomposition to zero provides the sequence satisfying topological requirements of (iii). Moreover, by Remark 2.9 any conformal class on the limiting space can be achieved, therefore, the proof of the lemma is complete.

□

3. Proof of Theorem 1.5

3.1. Case (i)

Let $\widetilde{\Sigma}_{\widetilde{\gamma}}$ be an orientable surface of genus $\widetilde{\gamma}$. By Lemma 2.16 there exists a sequence of conformal classes c_n such that the limiting space $\widehat{\Sigma_\infty}$ is a union of spheres. Since in the orientable case all geodesics are 2-sided, Theorem 2.11 implies

$$\lim_{n\to\infty} \Lambda_k(\widetilde{\Sigma}_\gamma, c_n) = \max_{\sum k_j = k} \sum \Lambda_{k_j}(\mathbb{S}^2).$$

We recall that by results of [**56**] one has $\Lambda_k(\mathbb{S}^2) = 8\pi k$. Therefore,

$$\widetilde{I}_k(\widetilde{\gamma}) \leqslant \lim_{n\to\infty} \Lambda_k(\widetilde{\Sigma}_{\widetilde{\gamma}}, c_n) = 8\pi k.$$

At the same time, by (1.3) one has $\widetilde{I}_k(\widetilde{\gamma}) \geqslant 8\pi k$.

3.2. Case (ii)

Let Σ_γ be a non-orientable surface of odd genus γ. By Lemma 2.16 there exists a sequence of conformal classes c_n such that all collapsing geodesics are 2-sided and the limiting space of $\widehat{\Sigma_\infty}$ is a union of spheres. Then the same argument as in Case (i) yields $I_k(\gamma) = 8\pi k$.

3.3. Case (iii)

Let Σ_γ be a non-orientable surface of odd genus γ. By Corollary 2.15, for any degenerate sequence c_n of conformal classes on Σ_γ either the limiting space $\widehat{\Sigma_\infty}$ contains non-orientable components of even genus or there exist 1-sided collapsing geodesics. We denote by $\Sigma_{\gamma'_i}$ the non-orientable components of $\widehat{\Sigma_\infty}$ of even genus γ'_i as well as projective planes with $\gamma'_i = 0$ for each collapsing 1-sided geodesic. Let M_j be the remaining components (orientable or non-orientable of odd genus). Then, Theorem 2.11 yields

$$I_k(\gamma) \leqslant \lim_{n\to\infty} \Lambda_k(\Sigma_\gamma, c_n) = \max\left(\sum \Lambda_{k_j}(M_j, c_\infty) + \sum \Lambda_{k'_j}(\Sigma_{\gamma'_i}, c_\infty)\right).$$

By Remark 2.9 one has that the conformal classes on the right hand side of the previous inequality range over all possible combinations of conformal classes on connected components of the limiting space. Therefore, taking the infimum over all possible degenerating sequences $\{c_n\}$ yields

$$I_k(\gamma) \leqslant \min_{\{\gamma'_j\},\{M_j\}} \max\left(\sum I_{k_j}(M_j) + \sum I_{k'_j}(\gamma'_j)\right), \tag{3.1}$$

where the minimum is taken over all possible topological types of the limiting space. Let K' be the set of indices k'_j and $|K'|$ denote the sum of all k'_j. Similarly, let Γ' be the set of genera γ'_j and $|\Gamma'|$ be the sum of γ'_j. Taking into account cases (i) and (ii) proved above, inequality (3.1) implies

$$I_k(\gamma) \leqslant \min_{\Gamma'} \max_{K',|K'|\leqslant k} \left(\sum I_{k'_j}(\gamma'_j) + 8\pi(k - |K'|)\right), \tag{3.2}$$

where the minimum is taken over all possible Γ' the limiting space can have. Lemma 2.16 implies that for all even $\gamma' < \gamma$ the sets $\Gamma' = \{\gamma'\}$ are possible. We claim that the minimum is actually attained on these one element sets. Indeed, assume Γ' contains two elements γ'_1 and γ'_2, then by inequality (1.3) for any k_1, k_2 one has $I_{k_1}(\gamma'_1) + I_{k_2}(\gamma'_2) \geqslant I_{k_1}(\gamma'_1) + 8\pi k_2$. Thus, inequality (3.2) becomes

$$I_k(\gamma) \leqslant \min_{\gamma'<\gamma} \max_{k'\leqslant k} (I_{k'}(\gamma') + 8\pi(k-k')).$$

Furthermore, by inequality (1.3) the maximum is achieved when $k' = k$. Therefore,

$$I_k(\gamma) = \min_{\gamma'<\gamma} I_k(\gamma').$$

Finally, since this inequality holds for all γ it is equivalent to having $I_k(\gamma) \leqslant I_k(\gamma-2)$ for all even $\gamma > 0$.

If inequality (1.5) is strict, then the minimizing sequence of conformal classes can not be degenerate. Therefore, it has to have a genuine conformal class on Σ_γ as an accumulation point. By continuity of $\Lambda_k(\Sigma_\gamma,c)$ in c, see [**11**], one has $I_k(\gamma) = \Lambda_k(\Sigma_\gamma,c)$.

3.4. Proof of Corollary 1.6

We start with the following proposition.

Proposition 3.1. *Let (M,g) be a closed Riemannian surface not diffeomorphic to the sphere $\mathbb{S}^2$. Then one has*

$$\Lambda_k(M,[g]) > \Lambda_k(\mathbb{S}^2) = 8\pi k.$$

Proof. It is proved in [**56**] that $\Lambda_k(\mathbb{S}^2) = 8\pi k$. Then, a combination of Theorem 1.3 and inequality (1.2) yields

$$\Lambda_k(M,[g]) \geqslant \Lambda_1(M,[g]) + 8\pi(k-1) > 8\pi k = \Lambda_k(\mathbb{S}^2).$$

$\square$

Now we are ready to prove the inequality $I_k(\gamma) > 8\pi k$. Assume the contrary. Since $I_k(\gamma) \geqslant 8\pi k$, it implies that $I_k(\gamma) = 8\pi k$. At the same time $I_k(0) > 8\pi k$. Therefore, there exists an even γ', $2 \leqslant \gamma' \leqslant \gamma$ such that $I_k(\gamma) = \ldots = I_k(\gamma') < I_k(\gamma'-2)$. As a result, there exists a conformal class c on $\Sigma_{\gamma'}$ such that $I_k(\gamma) = I_k(\gamma') = \Lambda_k(\Sigma_{\gamma'},c) > 8\pi k$ by Proposition 3.1.

4. Neumann eigenvalues

In this section we recall some results on conformal Neumann eigenvalues. The results of the present section are used repeatedly in Section 5.

4.1. Convergence of Neumann spectrum

Lemma 4.1. *Let (M,g) be a closed compact Riemannian manifold. Consider a finite collection $\{B_\epsilon(p_i)\}_{i=1}^l$ of geodesic balls of radius ϵ centred at some points $p_1,\ldots,p_l \in M$. Then for all $k \geqslant 0$ the Neumann eigenvalues $\lambda_k^N(M \setminus \cup_{i=1}^l B_\epsilon(p_i), g)$ converge to the eigenvalues $\lambda_k(M,g)$ as $\epsilon \to 0$.*

For the proof we refer the reader to the paper [**3**, Theorem 2].

Next, we recall the following statement.

Proposition 4.2. *Let M be a closed n-dimensional manifold and $\Omega \subset M$ be a smooth domain. Assume the sequence of Riemannian metrics g_m on M converges in C^∞-topology to the metric g. Then $\Lambda_k(M,[g_m]) \to \Lambda_k(M,[g])$. Similarly, if $h_m|_{\overline{\Omega}}$ converge to $g|_{\overline{\Omega}}$ in C^∞-topology, then $\Lambda_k^N(\Omega,[h_m|_{\overline{\Omega}}]) \to \Lambda_k^N(\Omega,[g|_{\overline{\Omega}}])$.*

Proof. We show the statement for closed manifolds. The case of domains is treated in the exactly same way.

Let $\varepsilon > 0$. Then for large enough m one has

$$\frac{1}{(1+\varepsilon)^2} f g_m(v,v) \leqslant f g(v,v) \leqslant (1+\varepsilon)^2 f g_m(v,v), \quad \forall v \in \Gamma(TM \setminus \{0\}),$$

where f is any positive smooth function on M. Then by [**19**, Proposition 3.3] one has

$$\frac{1}{(1+\varepsilon)^{2(n-1)}} \lambda_k(M,fg_m) \leqslant \lambda_k(M,fg) \leqslant (1+\varepsilon)^{2(n-1)} \lambda_k(M,fg_m).$$

At the same time

$$\frac{1}{(1+\varepsilon)^n} Vol(M,fg_m) \leqslant Vol(M,fg) \leqslant (1+\varepsilon)^n Vol(M,fg_m).$$

As a result,

$$\frac{1}{(1+\varepsilon)^{2n}} \bar\lambda_k(M,fg_m) \leqslant \bar\lambda_k(M,fg) \leqslant (1+\varepsilon)^{2n} \bar\lambda_k(M,fg_m).$$

Taking the supremum over all f yields

$$\frac{1}{(1+\varepsilon)^{2n}} \Lambda_k(M,[g_m]) \leqslant \Lambda_k(M,[g]) \leqslant (1+\varepsilon)^{2n} \Lambda_k(M,[g_m]).$$

Since it holds for any $\varepsilon > 0$ the proof is complete. ∎

4.2. Discontinuous metrics

Let (M,g) be a closed Riemannian manifold of dimension n. Consider a set of pairwise disjoint smooth domains $\{\Omega_i\}_{i=1}^s$ in M such that $M = \bigcup_{i=1}^s \overline{\Omega}_i$. Let us consider a class of discontinuous metrics on M defined as $\rho g \in [g]$, where $\rho|_{\Omega_i} = \rho_i \in C^\infty(\overline{\Omega}_i)$ are positive. The space of such functions ρ will be denoted as $C^\infty_+(M,\{\Omega_i\})$. If we do not require the components to be positive, we omit the subscript $+$.

The metric ρg is not smooth. The spectrum of the Laplacian $\Delta_{\rho g}$ is defined as the set of critical values of the Rayleigh quotient

$$R_{\rho g}[\varphi] = \frac{\displaystyle\int_M \rho^{\frac{n-2}{2}} |\nabla_g \varphi|_g^2 dv_g}{\displaystyle\int_M \rho^{\frac{n}{2}} \varphi^2 dv_g}.$$

Let n_i be outward pointing normal vector for $(\overline{\Omega}_i,\rho_i g)$. Then an eigenfunction u corresponding to the eigenvalue λ is continuous across $\partial\Omega$ and satisfies the following system

$$\begin{cases} \Delta_{\rho g} u = \lambda u & \text{on} \quad \bigcup \Omega_i, \\ \rho_i^{\frac{n-1}{2}} \dfrac{\partial u}{\partial n_i} + \rho_j^{\frac{n-1}{2}} \dfrac{\partial u}{\partial n_j} = 0 & \text{on} \quad \overline{\Omega}_i \cap \overline{\Omega}_j. \end{cases}$$

Let $C_b(M,\{\Omega_i\}) \subset C^0(M)$ be a subspace of $C^\infty(M,\{\Omega_i\})$ consisting of functions v satisfying the above boundary condition for eigenfunctions. Then

$$\lambda_k(M,\rho g) = \inf_{E_{k+1}} \sup_{\varphi \in E_{k+1}} R_{\rho g}[\varphi],$$

where E_{k+1} ranges over all $(k+1)$-dimensional subspaces of $C_b(M,\{\Omega_i\})$.

Let us introduce the following notation

$$\Lambda_k(M,\{\Omega_i\},[g]) = \sup\{\bar\lambda_k(\rho g) \mid \rho \in C^\infty_+(M,\{\Omega_i\})\},$$

where $\bar\lambda_k(\rho g)$ is the normalized k-th eigenvalue given by

$$\bar\lambda_k(\rho g) = \lambda_k(\rho g)||\rho||_{L^{\frac{n}{2}}(M,g)}.$$

Lemma 4.3. *Let (M,g) be a Riemannian manifold of dimension n. Consider a set of pairwise disjoint smooth domains $\{\Omega_i\}_{i=1}^s$ in M such that $M = \bigcup_{i=1}^s \overline{\Omega}_i$. Then one has*

$$\Lambda_k(M,\{\Omega_i\},[g]) = \Lambda_k(M,[g])$$

Proof. In the paper [**34**, Lemma 2] this lemma is proved for $k = 1$. The proof carries over to the case of arbitrary k. The only change is to redefine the set S from the original proof to be

$$S = \{u \in H^1(M,g) |\ u \perp_{L^2(M,\rho g)} E_0,\ldots, E_{k-1}, \int_M \rho^{\frac{n}{2}} u^2 dv_g = 1\},$$

where E_k is the eigenspace corresponding to the k-th eigenvalue of the metric ρg. We refer the reader to [**34**] for details.

□

Lemma 4.4. *Let (M,g) be a closed Riemannian manifold of dimension n. Consider a set of pairwise disjoint smooth domains $\{\Omega_i\}_{i=1}^s$ in M such that $M = \bigcup_{i=1}^s \overline{\Omega}_i$. Let $(\Omega,h) = \sqcup(\overline{\Omega}_i, g|_{\overline{\Omega}_i})$. Then for all $k \geqslant 0$ one has*

$$\Lambda_k(M,[g]) \geqslant \Lambda_k^N(\Omega,[h]).$$

If (M,g) is compact with non-empty boundary with (Ω, h) as above, then

$$\Lambda_k^N(M,[g]) \geqslant \Lambda_k^N(\Omega,[h]).$$

Proof. The proof is a combination of a classical Dirichlet-Neumann bracketing argument and Lemma 4.3. It remains the same whether M has boundary or not. Below, we assume that M is closed.

Let $h^m \in [h]$ be a maximizing sequence of metrics for $\Lambda_k^N(\Omega,[h])$. Let $g^m \in [g]$ be a discontinuous metric on M defined as $g|_{\Omega_i} = h_i$. Since the space of test functions for the Neumann eigenvalues $C^\infty(M,\{\Omega_i\})$ is larger than $C_b(M,\{\Omega_i\})$, the variational characterization of eigenvalues implies that for all k one has $\lambda_k(M,g^m) \geqslant \lambda^N(\Omega,h^m)$. Taking the limit and using the fact that $Vol(\Omega,h^m) = Vol(M,g^m)$ yields

$$\Lambda_k(M,\{\Omega_i\},[g]) \geqslant \Lambda_k^N(\Omega,[h]).$$

An application of Lemma 4.3 completes the proof. □

4.3. Neumann spectrum of a subdomain.

The present section is devoted to the proof of Proposition 1.7. The idea is to introduce a conformal factor that vanishes outside Ω. However, the conformal factors are not allowed to be equal to 0. To circumvent this difficulty one has to go through an approximation procedure which is carried out below.

Let us first remind the statement of Proposition 1.7. We state it in a slightly more general way.

Proposition 4.5. *Let (M,g) be a closed Riemannian manifold, $\Omega \subset M$ is a smooth subdomain. Then for all k one has*

$$\Lambda_k(M,[g]) \geqslant \Lambda_k^N(\Omega,[g|_{\overline{\Omega}}]).$$

If M is compact with non-empty boundary and $\Omega \subset M$ is a smooth domain, then for all k

$$\Lambda_k^N(M,[g]) \geqslant \Lambda_k^N(\Omega,[g|_{\overline{\Omega}}]).$$

The proof of the boundary case is identical to the closed case. For that reason we only present the closed case below.

We introduce the conformal factor ρ_δ, so that $\rho_\delta|_\Omega \equiv 1$ and $\rho_\delta|_{M\setminus\Omega} \equiv \delta$.

Lemma 4.6. *One has*

$$\liminf_{\delta\to 0} \lambda_k(\rho_\delta g) \geqslant \lambda_k^N(\Omega,g),$$

where $\lambda_k^N(\Omega,g)$ is the k-th Neumann eigenvalue of the domain (Ω,g).

For similar statements see [**18**, Theorem III.1] and [**14**, Theorem 2.1].

Let us first show how to deduce Proposition 4.5 from Lemma 4.6.

Proof of Proposition 4.5. Let $\{h_i \mid h_i \in [g|_\Omega]\}$ be a maximizing sequence of metrics for the functional $\Lambda_k^N(\Omega,[g])$, i.e.

$$\lim_{i\to\infty} \bar{\lambda}_k^N(\Omega, h_i) = \Lambda_k^N(\Omega, [g])$$

Let $h_i = f_i g|_\Omega$, where $f_i \in C_+^\infty(\bar{\Omega})$. We define the metric $\widetilde{h_i} = \widetilde{f}_i g$ on M, where $\widetilde{f}_i$ is any positive continuation of the function f_i into Ω^c. Then we consider the metric $\rho_\delta\widetilde{h_i}$, where as before

$$\rho_\delta = \begin{cases} 1 & \text{in } \Omega, \\ \delta & \text{in } M \setminus \Omega. \end{cases}$$

By Lemma 4.6 we then have

$$\liminf_{\delta\to 0} \lambda_k(\rho_\delta\widetilde{h_i}) \geqslant \lambda_k^N(\Omega, h_i).$$

At the same time, $Vol(M,\rho_\delta\widetilde{h_i}) \to Vol(\Omega,h_i)$. Then, by Lemma 4.3 one obtains

$$\Lambda_k(M, [g]) = \Lambda_k(M,\{\Omega,M \setminus \Omega\},[g]) \geqslant \liminf_{\delta\to 0} \bar{\lambda}_k(\rho_\delta\widetilde{h_i}) \geqslant \bar{\lambda}_k^N(\Omega, h_i).$$

Taking the limit as $i \to \infty$ yields,

$$\Lambda_k(M, [g]) \geqslant \Lambda_k^N(\Omega, [g]).$$

$\square$

Proof of Lemma 4.6. The proof below is essentially the proof in [**26**, Section 2, Step 2, Step 3] with details added. We denote $M \setminus \Omega$ by Ω^c. Let

$$\mathcal{H}_1 := \{\varphi \in H^1(M,g) \mid (\Delta\varphi)_{|\Omega^c} = 0\}$$

and

$$\mathcal{H}_2 := \{\varphi \in H^1(M,g) \mid \varphi \in H^1_0(\Omega^c,g),\, \varphi_{|\Omega} = 0\}.$$

Claim 1. One has the following decomposition of $H^1(M,g)$

$$H^1(M,g) = \mathcal{H}_1 \oplus \mathcal{H}_2$$

into the sum of closed subspaces. Moreover for any $\delta > 0$ one has

$$\int_M \langle \nabla\varphi, \nabla\psi \rangle_{\rho_\delta g} dv_{\rho_\delta g} = 0, \forall \varphi \in \mathcal{H}_1, \psi \in \mathcal{H}_2,$$

where as before $\rho_\delta|_\Omega \equiv 1$ and $\rho_\delta|_{M\setminus\Omega} \equiv \delta$.

Proof. Since $H^1_0(\Omega,g)$ is complete we immediately conclude that $\mathcal{H}_2$ is a closed subspace of $H^1(M,g)$.

We show that the space $\mathcal{H}_1$ is also closed. Consider the mapping:

$$T\colon H^1(M,g) \to \mathcal{H}_2,$$

defined as

$$T\varphi = \begin{cases} 0 & \text{in } \Omega, \\ \widehat{\varphi} - \varphi & \text{in } \Omega^c, \end{cases}$$

where $\widehat{\varphi}$ is the harmonic extension into Ω^c of the restriction $\varphi|_{\partial\Omega}$. Since $\mathcal{H}_1 = \ker T$, it is sufficient to show that T is continuous.

We have $T = T' - Id$, where

$$T'\varphi = \begin{cases} \varphi & \text{in } \Omega, \\ \widehat{\varphi} & \text{in } \Omega^c \end{cases}$$

and Id is the identity mapping. We recall the following estimate [**104**, Proposition 1.7, p.360]

$$||\widehat{\varphi}||_{H^1(\Omega^c,g)} \leqslant C||\varphi|_{\partial\Omega}||_{H^{1/2}(\partial\Omega,g)}.$$

In the following, the letter C denotes any constant depending only on (M,g) and Ω. Its exact value could differ from line to line. By the Trace Embedding Theorem one has

$$||\varphi_{|\partial\Omega}||_{H^{1/2}(\partial\Omega,g)} \leqslant C||\varphi||_{H^1(\Omega,g)}.$$

Finally, we have

$$||\varphi||_{H^1(\Omega,g)} \leqslant ||\varphi||_{H^1(M,g)}.$$

All the above implies

$$||\widehat{\varphi}||_{H^1(\Omega^c,g)} \leqslant C||\varphi||_{H^1(M,g)}.$$

Therefore, one has

$$||T'\varphi||^2_{H^1(M,g)} = ||T'\varphi||^2_{H^1(\Omega,g)} + ||T'\varphi||^2_{H^1(\Omega^c,g)} = ||\varphi||^2_{H^1(\Omega,g)} + ||\widehat{\varphi}||^2_{H^1(\Omega^c,g)} \leqslant$$
$$\leqslant ||\varphi||^2_{H^1(M,g)} + ||\widehat{\varphi}||^2_{H^1(\Omega^c,g)} \leqslant ||\varphi||^2_{H^1(M,g)} + C^2||\varphi||^2_{H^1(M,g)} \leqslant C||\varphi||^2_{H^1(M,g)},$$

which completes the proof that T is continuous.

Finally, we prove that for any $\delta > 0$ one has

$$\int_M \langle \nabla\varphi, \nabla\psi\rangle_{\rho_\delta g} dv_{\rho_\delta g} = 0, \forall \varphi \in \mathcal{H}_1, \psi \in \mathcal{H}_2.$$

Indeed,

$$\int_M \langle \nabla\varphi, \nabla\psi\rangle_{\rho_\delta g} dv_{\rho_\delta g} = \int_\Omega \langle \nabla\varphi, \nabla\psi\rangle_{\rho_\delta g} dv_{\rho_\delta g} + \int_{\Omega^c} \langle \nabla\varphi, \nabla\psi\rangle_{\rho_\delta g} dv_{\rho_\delta g} =$$
$$= \int_\Omega \langle \nabla\varphi, \nabla\psi\rangle_g dv_g + \delta^{\frac{n-2}{2}} \int_{\Omega^c} \langle \nabla\varphi, \nabla\psi\rangle_g dv_g =$$
$$= \int_\Omega \Delta\varphi\psi \; dv_g + \delta^{\frac{n-2}{2}} \int_{\Omega^c} \Delta\varphi\psi \; dv_g = 0,$$

since $\psi_{|\partial\Omega} = 0$.

□

For a function $\varphi \in H^1(M,g)$ we fix its decomposition $\varphi_1 + \varphi_2$ with

$$\varphi_1 = \begin{cases} \varphi & \text{in } \Omega, \\ \widehat{\varphi_{|\Omega}} & \text{in } \Omega^c \end{cases}$$

and $\varphi_2 = \varphi_1 - \varphi$.

For the sake of simplicity we use the symbols λ_k^δ for $\lambda_k(\rho_\delta g)$, g_δ for $\rho_\delta g$ and R_δ for the Rayleigh quotient

$$R_\delta[\varphi] = \frac{\int_M |\nabla\varphi|^2_{g_\delta} dv_{g_\delta}}{\int_M \varphi^2 dv_{g_\delta}}.$$

Claim 2. There exists a constant $C_k > 0$ such that $\lambda_k^\delta \leqslant C_k$.

Proof. Theorem 1.1 implies that there exists a constant $C(k) > 0$ such that

$$\Lambda_k(M, [g]) \leqslant C(k).$$

By Lemma 4.3 for every δ one has

$$\lambda_k^\delta Vol^{\frac{2}{n}}(M,g_\delta) \leqslant \Lambda_k(M, [g]) \leqslant C(k).$$

Therefore,

$$\lambda_k^\delta \leqslant \frac{C(k)}{Vol^{\frac{2}{n}}(M,g_\delta)} = \frac{C(k)}{(Vol(\Omega,g)+\delta^{\frac{n}{2}}Vol(\Omega^c,g))^{\frac{2}{n}}} \leqslant \frac{C(k)}{(Vol(\Omega,g))^{\frac{2}{n}}} = C_k$$

□

Let W_k be the set of $k+1$-dimensional subspaces of $H^1(M,g_\delta)$ satisfying the condition that $R_\delta|_{W_k} \leqslant C_k$. We remark that according to Claim 2 the space spanned by the first $k+1$ eigenfunctions is in W_k, i.e. $W_k \neq \varnothing$.

Claim 3. For every $\varphi \in V \in W_k$ there exists a constant $C > 0$ such that

$$\int_{\Omega^c} \varphi_2^2 \; dv_{g_\delta} \leqslant C\delta \int_M \varphi^2 dv_{g_\delta}.$$

Proof. By Claim 1 one has

$$\int_M \langle \nabla\varphi_1, \nabla\varphi_2\rangle_{g_\delta} dv_{g_\delta} = 0.$$

Further, since $\varphi \in V \in W_k$ we have

$$C_k \geqslant R_\delta[\varphi] = \frac{\displaystyle\int_M |\nabla\varphi|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_M \varphi^2 dv_{g_\delta}} = \frac{\displaystyle\int_M |\nabla\varphi_1|^2_{g_\delta} dv_{g_\delta} + \int_M |\nabla\varphi_2|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_M \varphi^2 dv_{g_\delta}} \geqslant$$

$$\geqslant \frac{\displaystyle\int_{\Omega^c} |\nabla\varphi_2|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_M \varphi^2 dv_{g_\delta}} = \frac{1}{\delta}\frac{\displaystyle\int_{\Omega^c} |\nabla\varphi_2|^2_{g} dv_{g}}{\displaystyle\int_M \varphi_2^2 dv_{g}} \frac{||\varphi_2||^2_{L^2(\Omega^c,g_\delta)}}{||\varphi||^2_{L^2(M,g_\delta)}} \geqslant \frac{\lambda_1^D(\Omega^c,g)}{\delta}\frac{||\varphi_2||^2_{L^2(\Omega^c,g_\delta)}}{||\varphi||^2_{L^2(M,g_\delta)}},$$

where $\lambda_1^D(\Omega^c,g)$ is the first non-zero Dirichlet eigenvalue of (Ω^c,g). □

Claim 4. For every $\varphi \in V \in W_k$ and for every sufficiently small δ there exists a constant $C > 0$ such that

$$\int_M \varphi^2 \; dv_{g_\delta} \leqslant (1 + C\sqrt{\delta}) \int_M \varphi_1^2 dv_{g_\delta}.$$

Proof. One has

$$||\varphi||^2_{L^2(M,g_\delta)} = \int_{\Omega^c} (\varphi_1+\varphi_2)^2 dv_{g_\delta} + \int_\Omega \varphi_1^2 dv_{g_\delta} \leqslant \Big(1+\frac{1}{\varepsilon}\Big)\int_M \varphi_2^2 dv_{g_\delta} + (1+\varepsilon)\int_M \varphi_1^2 dv_{g_\delta},$$

for every $\varepsilon > 0$. Applying Claim 3 we obtain

$$||\varphi||^2_{L^2(g_\delta)} \leqslant C\delta\Big(1+\frac{1}{\varepsilon}\Big)\int_M \varphi^2 dv_{g_\delta} + (1+\varepsilon)\int_M \varphi_1^2 dv_{g_\delta},$$

and hence,

$$\Big(1 - C\delta\Big(1+\frac{1}{\varepsilon}\Big)\Big)||\varphi||^2_{L^2(M,g_\delta)} \leqslant (1+\varepsilon)||\varphi_1||^2_{L^2(M,g_\delta)}.$$

Choosing $\varepsilon = \sqrt{\delta}$ completes the proof. □

Claim 5. For every $\varphi \in V \in W_k$ and for every sufficiently small δ there exists a constant $C > 0$ such that

$$\int_{\Omega^c} \varphi_1^2 \, dv_g \leqslant C \int_\Omega \varphi_1^2 dv_g.$$

Proof. By the Sobolev Embedding Theorem one has

$$||\varphi_1||_{L^2(\Omega^c,g)} \leqslant C||\varphi_1||_{H^1(\Omega^c,g)}.$$

Again by [**104**, Proposition 1.7, p.360]) one has

$$||\varphi_1||_{H^1(\Omega^c,g)} \leqslant C||\varphi|_{\partial\Omega}||_{H^{1/2}(\partial\Omega,g)}.$$

By the Trace Embedding Theorem one has

$$||\varphi_{|\partial\Omega}||_{H^{1/2}(\partial\Omega,g)} \leqslant C||\varphi_1||_{H^1(\Omega,g)}.$$

Altogether

$$||\varphi_1||_{L^2(\Omega^c,g)} \leqslant C||\varphi_1||_{H^1(\Omega,g)}. \tag{4.1}$$

Further, since $\varphi \in V \in W_k$ and $\int_M \langle \nabla\varphi_1, \nabla\varphi_2\rangle_{g_\delta} dv_{g_\delta} = 0$ one has

$$C_k \geqslant R_\delta[\varphi] = \frac{\int_M |\nabla\varphi|^2_{g_\delta} dv_{g_\delta}}{\int_M \varphi^2 dv_{g_\delta}} = \frac{\int_M |\nabla\varphi_1|^2_{g_\delta} dv_{g_\delta} + \int_M |\nabla\varphi_2|^2_{g_\delta} dv_{g_\delta}}{\int_M \varphi^2 dv_{g_\delta}} \geqslant \frac{\int_\Omega |\nabla\varphi_1|^2_g dv_g}{\int_M \varphi^2 dv_{g_\delta}},$$

hence,

$$\int_\Omega |\nabla\varphi_1|^2_g \, dv_g \leqslant C_k \int_M \varphi^2 dv_{g_\delta},$$

and by Claim 4 one gets

$$\int_\Omega |\nabla\varphi_1|^2_g \, dv_g \leqslant C_k(1 + C\sqrt{\delta})||\varphi_1||^2_{L^2(M,g_\delta)} =$$
$$= C_k(1 + C\sqrt{\delta})(||\varphi_1||^2_{L^2(\Omega,g)} + \delta^{n/2}||\varphi_1||^2_{L^2(\Omega^c,g)}).$$

Plugging the latter in (4.1) we obtain

$$||\varphi_1||^2_{L^2(\Omega^c,g)} \leqslant C||\varphi_1||^2_{H^1(\Omega,g)} = C(||\varphi_1||^2_{L^2(\Omega,g)} + ||\nabla\varphi_1||^2_{L^2(\Omega,g)}) \leqslant$$
$$\leqslant C\Big(||\varphi_1||^2_{L^2(\Omega,g)} + C_k(1 + C\sqrt{\delta})(||\varphi_1||^2_{L^2(\Omega,g)} + \delta^{n/2}||\varphi_1||^2_{L^2(\Omega^c,g)})\Big).$$

Rearranging the terms yields the required inequality. $\square$

By Claim 4 for every $\varphi \in V \in W_k$ and $\int_M \langle \nabla\varphi_1, \nabla\varphi_2\rangle_{g_\delta} dv_{g_\delta} = 0$ one has

$$R_\delta[\varphi] = \frac{\displaystyle\int_M |\nabla\varphi|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_M \varphi^2 dv_{g_\delta}} = \frac{\displaystyle\int_M |\nabla\varphi_1|^2_{g_\delta} dv_{g_\delta} + \int_M |\nabla\varphi_2|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_M \varphi^2 dv_{g_\delta}} \geqslant$$

$$\geqslant \frac{1}{1+C\sqrt{\delta}} \frac{\displaystyle\int_M |\nabla\varphi_1|^2_{g_\delta} dv_{g_\delta} + \int_M |\nabla\varphi_2|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_M \varphi_1^2 dv_{g_\delta}} \geqslant$$

$$\geqslant \frac{1}{1+C\sqrt{\delta}} \frac{\displaystyle\int_M |\nabla\varphi_1|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_M \varphi_1^2 dv_{g_\delta}} = \frac{1}{1+C\sqrt{\delta}} \frac{\displaystyle\int_M |\nabla\varphi_1|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_\Omega \varphi_1^2 dv_g + \delta^{\frac{n}{2}} \int_{\Omega^c} \varphi_1^2 dv_g}.$$

By Claim 5 we then have

$$R_\delta[\varphi] \geqslant \frac{1}{(1+\delta^{\frac{n}{2}}C)(1+C\sqrt{\delta})} \frac{\displaystyle\int_M |\nabla\varphi_1|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_\Omega \varphi_1^2 dv_g} \geqslant$$

$$\geqslant \frac{1}{(1+\delta^{\frac{n}{2}}C)(1+C\sqrt{\delta})} \frac{\displaystyle\int_\Omega |\nabla\varphi_1|^2_g dv_g}{\displaystyle\int_\Omega \varphi_1^2 dv_g} = \frac{1}{(1+\delta^{\frac{n}{2}}C)(1+C\sqrt{\delta})} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2_g dv_g}{\displaystyle\int_\Omega \varphi^2 dv_g} \geqslant$$

$$\geqslant \frac{1}{(1+\delta^{\frac{n}{2}}C)(1+C\sqrt{\delta})} R^N_{(\Omega,g)}[\varphi|_\Omega],$$

where $R^N_{(\Omega,g)}$ denotes the Rayleigh quotient for the Neumann problem in the domain (Ω,g). Let $V = \mathrm{span}\langle\psi_0,\ldots,\psi_k\rangle$, where ψ_i is in the i-th eigenspace of (M,g_δ). Then

$$\begin{aligned}\lambda_k^\delta = \max_{\varphi\in V} R_\delta[\varphi] &\geqslant \frac{1}{(1+\delta^{\frac{n}{2}}C)(1+C\sqrt{\delta})} \max_{\varphi\in V} R^N_{(\Omega,g)}[\varphi|_\Omega] \geqslant \\ &\geqslant \frac{1}{(1+\delta^{\frac{n}{2}}C)(1+C\sqrt{\delta})} \lambda_k^N(\Omega,g),\end{aligned} \tag{4.2}$$

since by unique continuation the restriction to Ω of the functions ψ_i form the space of the same dimension. Taking the $\liminf$ as $\delta \to 0$ in (4.2) competes the proof. □

Using Proposition 2.6 one gets the following corollary.

Corollary 4.7. *Let (M,g) be a closed compact Riemannian manifold. Consider a sequence $\{K_n\}$ of smooth domains $K_n \subset M$ such that*

- $K_r \subset K_s$ $\forall r > s$;
- $\cap_n K_n = \{p_1,\ldots,p_l\}$ *for some points* $p_1,\ldots,p_l \in M$.

Then one has

$$\lim_{n\to\infty} \Lambda_k^N(M \setminus K_n, [g]) = \Lambda_k(M,[g]).$$

Proof. Proposition 4.5 implies that

$$\limsup_{n\to\infty} \Lambda_k^N(M \setminus K_n, [g]) \leqslant \Lambda_k(M, [g]).$$

It remains to show that

$$\Lambda_k(M, [g]) \leqslant \liminf_{n\to\infty} \Lambda_k^N(M \setminus K_n, [g]).$$

Let g^m be a maximizing sequence for the functional $\Lambda_k(M, [g])$. Then for a fixed m we consider geodesic balls $B_{\epsilon_n}(p_i)$ of radius $\epsilon_n \to 0$ in metric g^m centred at the points $p_1, \dots, p_l \in M$ such that $K_n \subset \cup_{i=1}^l B_{\epsilon_n}(p_i)$. Then $M \setminus \cup_{i=1}^l B_{\epsilon_n}(p_i) \subset M \setminus K_n$ and Proposition 4.5 implies that

$$\Lambda_k^N(M \setminus K_n,[g]) \geqslant \Lambda_k^N(M \setminus \cup_{i=1}^l B_{\epsilon_n}(p_i),[g]) \geqslant \bar\lambda_k^N(M \setminus \cup_{i=1}^l B_{\epsilon_n}(p_i),g^m). \tag{4.3}$$

Note that $Vol(M \setminus \cup_{i=1}^l B_{\epsilon_n}(p_i),g^m) \to Vol(M,g^m)$ as $n \to \infty$ and by Lemma 4.1 one has $\lambda_k^N(M \setminus \cup_{i=1}^l B_{\epsilon_n}(p_i),g^m) \to \lambda_k(M,g^m)$. Hence, $\bar\lambda_k^N(M \setminus \cup_{i=1}^l B_{\epsilon_n}(p_i),g^m) \to \bar\lambda_k(M,g^m)$ as $n \to \infty$. Taking $\liminf_{n\to\infty}$ in (7.11) one then gets

$$\liminf_{n\to\infty} \Lambda_k^N(M \setminus K_n,[g]) \geqslant \bar\lambda_k(M,g^m).$$

Passing to the limit as $m \to \infty$ completes the proof. $\square$

4.4. Disconnected manifolds.

Lemma 4.8. *Let $(\Omega,g) = \sqcup_{i=1}^s(\Omega_i,g_i)$ be a disjoint union of Riemannian manifolds of dimension n with smooth boundary. Then for all $k > 0$ one has*

$$\Lambda_k^N(\Omega,[g])^{\frac{n}{2}} = \max_{\sum_{i=1}^s k_i=k,\ k_i>0} \sum_{i=1}^s \Lambda_{k_i}^N(\Omega_i,[g_i])^{\frac{n}{2}}.$$

Similarly, if $(M,g) = \sqcup_{i=1}^s(M_i,g_i)$ is a disjoint union of closed Riemannian manifolds of dimension n, then one has

$$\Lambda_k(M,[g])^{\frac{n}{2}} = \max_{\sum_{i=1}^s k_i=k,\ k_i>0} \sum_{i=1}^s \Lambda_{k_i}(M_i,[g_i])^{\frac{n}{2}}.$$

Proof. The proof is reminiscent of the argument due to Wolf and Keller [**108**]. The differences between the proofs of two equalities are cosmetic, we only present the proof of the first equality.

Inequality $\geqslant$.

Fix the indices $k_i > 0$ satisfying $\sum k_i = k$. Let $\{g_i^m\}$ be a maximizing sequence of metrics such that $\bar\lambda_{k_i}^N(\Omega_i,g_i^m) \to \Lambda_{k_i}^N(\Omega_i,[g_i])$. Up to a rescaling one can assume that $\lambda_{k_i}^N(\Omega_i, g_i^m) =$

$\Lambda_k^N(\Omega,[g])$. Then, one has

$$Vol(\Omega_i, g_i^m) \to \frac{\Lambda_{k_i}^N(\Omega_i,[g_i])^{\frac{n}{2}}}{\Lambda_k^N(\Omega,[g])^{\frac{n}{2}}}$$

Consider a sequence of metrics $\{g^m\}$ on Ω defined as $g^m|_{\Omega_i} = g_i^m$. Since the spectrum of disjoint union is the union of spectra of each component, then for large enough m one has that $\lambda_k^N(\Omega,g^m) = \Lambda_k^N(\Omega,[g])$. At the same time, by definition of $\Lambda_k^N(\Omega,[g])$ one has

$$\Lambda_k^N(\Omega,[g])Vol(\Omega,g^m)^{\frac{2}{n}} = \lambda_k^N(\Omega,g^m)Vol(\Omega,g^m)^{\frac{2}{n}} \leqslant \Lambda_k^N(\Omega,[g]),$$

i.e. $Vol(\Omega,g^m) \leqslant 1$. Therefore, one obtains

$$1 \geqslant Vol(\Omega,g^m) = \sum_i Vol(\Omega_i,g_i^m) \to \frac{\sum_i \Lambda_{k_i}^N(\Omega_i,[g_i])^{\frac{n}{2}}}{\Lambda_k^N(\Omega,[g])^{\frac{n}{2}}}.$$

Passing to the limit $m \to \infty$ yields the inequality.

Inequality $\leqslant$.

Assume the contrary, i.e.

$$\Lambda_k^N(\Omega,[g])^{\frac{n}{2}} > \max_{\sum_{i=1}^s k_i = k,\ k_i > 0} \sum_{i=1}^s \Lambda_{k_i}^N(\Omega_i,[g_i])^{\frac{n}{2}}. \tag{4.4}$$

Let $\{g^m\}$ be a maximizing sequence of metrics of volume 1 such that $\lambda_k^N(\Omega,g^m) \to \Lambda_k^N(\Omega,[g])$. Let g_i^m be a restriction of g^m to Ω_i. Further, let d_i^m be the largest number such that $\lambda_{d_i^m}^N(\Omega_i,g_i^m) < \Lambda_k^N(\Omega,[g])$ and $\limsup_{m\to\infty} \lambda_{d_i^m}^N(\Omega_i,g_i^m) < \Lambda_k^N(\Omega,[g])$ and V_i^m be $Vol(\Omega_i,g_i^m)$. Then one has $d_i^m \leqslant k$ and $V_i^m \leqslant 1$. Therefore, up to a choice of a subsequence one can assume that $d_i^m = d_i$ does not depend on m and $V_i^m \to V_i$ as $m \to \infty$.

We claim that $\sum_i (d_i + 1) \geqslant k+1$. Otherwise, by (4.4) and definition of d_i one has

$$\Lambda_k^N(\Omega,[g])^{\frac{n}{2}} \sum_i V_i \leqslant \sum_i \limsup_{m\to\infty} \bar\lambda_{d_i+1}^N(\Omega_i,g_i^m)^{\frac{n}{2}} \leqslant \sum_i \Lambda_{d_i+1}^N(\Omega_i,[g])^{\frac{n}{2}} < \Lambda_k^N(\Omega,[g])^{\frac{n}{2}}.$$

Since g^m are of unit volume, one has $\sum_i V_i = 1$. Thus, one arrives at $\Lambda_k^N(\Omega,[g])^{\frac{n}{2}} < \Lambda_k^N(\Omega,[g])^{\frac{n}{2}}$, which is a contradiction.

Therefore, one has $\sum(d_i+1) \geqslant k+1$. Since the spectrum of a union is a union of spectra, one has $\lambda_k^N(\Omega,g^m) \in \bigcup_i \{\lambda_0(\Omega_i,g_i^m),\ldots,\lambda_{d_i}(\Omega_i,g_i^m)\}$, i.e.

$$\Lambda_k^N(\Omega,g) = \limsup_{m\to\infty} \lambda_k^N(\Omega,g^m) \leqslant \max_i \limsup_{m\to\infty} \lambda_{d_i}(\Omega_i,g_i^m) < \Lambda_k^N(\Omega,[g]).$$

Since g^m are of unit volume we arrive at a contradiction. $\square$

Finally, as a corollary of Lemma 4.4, Proposition 4.5 and Lemma 4.8 one obtains.

Lemma 4.9. *Let (M,g) be a closed Riemannian manifold of dimension n. Consider a set of pairwise disjoint smooth domains $\{\Omega_i\}_{i=1}^s$ in M such that $M = \bigcup_{i=1}^s \overline{\Omega}_i$. Then one has*

$$\Lambda_k(M,[g])^{\frac{n}{2}} \geqslant \max_{\sum_{i=1}^s k_i = k,\ k_i \geqslant 0} \sum_{i=1}^s \Lambda_{k_i}^N(\Omega_i,[g])^{\frac{n}{2}}.$$

If M is compact with non-empty boundary, then one has

$$\Lambda_k^N(M,[g])^{\frac{n}{2}} \geqslant \max_{\sum_{i=1}^s k_i=k,\ k_i\geqslant 0} \sum_{i=1}^s \Lambda_{k_i}^N(\Omega_i,[g])^{\frac{n}{2}}.$$

PROOF. Once again, we only give a proof for the closed case.

Fix indices $k_i \geqslant 0$ such that $\sum_{i=1}^s k_i = k$. Let $I = \{i \,|\, k_i > 0\}$ and set $\Omega_1 = \cup_{i\in I}\overline{\Omega}_i \subset M$, $(\Omega_2,h) = \sqcup_{i\in I}(\overline{\Omega}_i,g_{\overline{\Omega}_i})$. Applying in order: Proposition 4.5, Lemma 4.4 and Lemma 4.8, one obtains

$$\Lambda_k(M,[g]) \geqslant \Lambda_k^N(\Omega_1,[g]) \geqslant \Lambda_k(\Omega_2,[h]) \geqslant \sum_{i\in I} \Lambda_{k_i}^N(\Omega_i,[g])^{\frac{n}{2}} = \sum_{i=1}^s \Lambda_{k_i}^N(\Omega_i,[g])^{\frac{n}{2}},$$

where in the last equality we used that $\Lambda_0(\Omega_j,[g]) = 0$ for any j. $\square$

5. Proof of Theorem 2.11

We remind the reader that as $n \to \infty$ one has s 1-sided geodesics and $\widetilde{s}$ 2-sided geodesics collapse and the canonical representative metric $h_n \in c_n$ is hyperbolic if $\chi(\Sigma_\gamma) < 0$ and is flat if $\chi(\Sigma_\gamma) = 0$. We start with the hyperbolic case and discuss the flat case at the end of the section.

We introduce the following notations

- $\mathcal{C}_i^n$ for collars of 1-sided collapsing geodesics, $i = 1,\dots,s$. Their width is denoted by w_i^n
- $\widetilde{\mathcal{C}}_i^n$ for collars of 2-sided collapsing geodesics, $i = 1,\dots,\widetilde{s}$. Their width is denoted by $\widetilde{w}_i^n$
- M_j^n for a connected component of $M \setminus (\cup_{i=1}^s \mathcal{C}_i^n \bigcup \cup_{i=1}^{\widetilde{s}} \widetilde{\mathcal{C}}_i^n)$
- for $-\widetilde{w}_i^n \leqslant a \leqslant b \leqslant \widetilde{w}_i^n$, we denote $\widetilde{\mathcal{C}}_i^n(a,b) \subset \widetilde{\mathcal{C}}_i^n$ the subset $\{(t,\theta)\,|a \leqslant t \leqslant b\}$
- for $0 \leqslant a \leqslant b \leqslant w_i^n$, we denote $\mathcal{C}_i^n(a,b) \subset \mathcal{C}_i^n$ the subset

  $$\{(t,\theta)\,|a \leqslant t \leqslant b\} \cup \{(t,\theta)\,|-b \leqslant t \leqslant -a\}/\sim .$$

  It is a Möbius band if $a = 0$ and cylinder otherwise.
- Let $\alpha^n = \cup_{i=1}^s \alpha_i^n \bigcup \cup_{i=1}^{\widetilde{s}}\{\alpha_{j,-}^n,\alpha_{j,+}^n\}$, where $0 \leqslant \alpha_i^n \leqslant w_i^n$ and $-\widetilde{w}_i^n \leqslant \alpha_{i,-}^n \leqslant \alpha_{i,+}^n \leqslant \widetilde{w}_i^n$. We denote by $M_j^n(\alpha^n)$ the connected component of

  $$M \setminus \Big(\cup_{i=1}^s \mathcal{C}_i^n(0,\alpha_i^n) \bigcup \cup_{i=1}^{\tilde{s}} \widetilde{\mathcal{C}}_i^n(\alpha_{i,-}^n,\alpha_{i,+}^n)\Big)$$

  which contains M_j^n;
- $a_n \ll b_n$ for two sequences $\{a_n\}$ and $\{b_n\}$ satisfying $a_n, b_n \to +\infty$ and $\frac{a_n}{b_n} \to 0$ as $n \to \infty$.

5.1. Inequality $\geqslant$.

We start with proving the inequality

$$\liminf_{n\to\infty} \Lambda_k(M,c_n) \geqslant \max\Big(\sum_{i=1}^{\widetilde m} \Lambda_{\widetilde{k}_i}(\widetilde\Sigma_{\widetilde\gamma_i}, c_\infty) + \sum_{i=1}^{m} \Lambda_{k_i}(\Sigma_{\gamma_i},c_\infty) + \sum_{i=1}^{\widetilde s} \Lambda_{\widetilde{r}_i}(\mathbb{S}^2) + \sum_{i=1}^{s} \Lambda_{r_i}(\mathbb{RP}^2)\Big), \tag{5.1}$$

Consider the domains $\mathcal{C}^n_i(0,\alpha^n_i)$ for $1 \leqslant i \leqslant s$, $\widetilde{\mathcal{C}}^n_i(\alpha^n_{i,-},\alpha^n_{i,+})$ for $1 \leqslant i \leqslant \tilde s$, where $w^n_i - \alpha^n_i \ll w^n_i$, $\alpha^n_i \to \infty$ and $\widetilde w^n_i - \alpha^n_{i,\pm} \ll \widetilde w^n_i$, $\alpha^n_{i,\pm} \to \infty$ and the domain $M^n_j(\alpha^n)$. By Lemma 4.9 we have

$$\Lambda_k(M,c_n) \geqslant \max\Big(\sum_{i=1}^{s} \Lambda^N_{r_i}(\mathcal{C}^n_i(0,\alpha^n_i),c_n) + \sum_{i=1}^{\tilde s} \Lambda^N_{\tilde r_i}(\widetilde{\mathcal{C}}^n_i(\alpha^n_{i,-},\alpha^n_{i,+}),c_n) + \sum_{j=1}^{m+\widetilde m} \Lambda^N_{k_j}(M^n_j(\alpha^n),c_n)\Big). \tag{5.2}$$

For $1 \leqslant i \leqslant \tilde s$ we define the conformal maps $\widetilde\Psi^n_i\colon (\widetilde{\mathcal{C}}^n_i(\alpha^n_{i,-},\alpha^n_{i,+}),c_n) \to (\mathbb{S}^2,[g_{can}])$ as

$$\widetilde\Psi^n_i(t,\theta) = \frac{1}{e^{2t}+1}(2e^t\cos\theta, 2e^t\sin\theta, e^{2t}-1).$$

For $1 \leqslant i \leqslant s$ we define the conformal maps $\Psi^n_i\colon (\mathcal{C}^n_i(0,\alpha^n_i),c_n) \to (\mathbb{RP}^2,[g_{can}])$ as the maps, such that their lift to orientable double covers is given by the same formula as $\widetilde\Psi^n_i$. Finally, we take a restriction of a diffeomorphism Ψ_n^{-1} given by Proposition 2.6 to obtain a conformal map $\check\Psi^n_j\colon (M^n_j(\alpha^n),c_n) \to (M_\infty,\Psi^*_n c_n)$.

Let $\Omega^n_i \subset \mathbb{RP}^2$, $\widetilde\Omega^n_i \subset \mathbb{S}^2$ and $\check\Omega^n_j \subset M_\infty$ be the the images of Ψ^n_i, $\widetilde\Psi^n_i$ and $\check\Psi^n_j$ respectively. Since $\alpha^n_i, \alpha^n_{i,\pm} \to \infty$, the domains Ω^n_i and $\widetilde\Omega^n_i$ exhaust $\mathbb{RP}^2$ and $\mathbb{S}^2$ respectively. The corresponding statement for $\check\Omega^n_j$ is the content of the following lemma.

Lemma 5.1. *Let M^∞_j be the connected component $\check\Psi^n_j(M^n_j) \subset M_\infty$. Then the domains $\check\Omega^n_j$ exhaust M^∞_j.*

Proof. Let $M_\infty = M^\infty_{\geqslant\varepsilon} \cup M^\infty_{<\varepsilon}$ be an ε-thick-thin decomposition of (M_∞,h_∞). For a sufficiently small $0<\varepsilon<arcsinh(1)$ the ε-thin part $M^\infty_{<\varepsilon}$ is nothing but subcollars of cusps (see [**45**, Proposition IV.4.2]). For the surface (M,h_n) we set l^n_i for the length of the i-th 1-sided pinching geodesic and $\tilde l^n_j$ for the length of the j-th 2-sided pinching geodesic, where as before $i=1,\ldots,s$ and $j=1,\ldots,\tilde s$. Consider the diffeomorphism $(\Psi^n)^{-1}\colon M \to M_\infty$. From [**110**, formula (4.12)] it follows that for a fixed ε and for all $\varepsilon_1<\varepsilon$ there exists a number N_1 such that for all $n > N_1$ one has

$$\cup_{j=1}^{\tilde s}\widetilde{\mathcal{C}}^n_j(\tilde\beta^n_j(\varepsilon_1),\tilde\beta^n_j(\varepsilon_1)) \bigcup \cup_{i=1}^{s}\mathcal{C}^n_i(0,\beta^n_i(\varepsilon_1)) \subseteq \Psi^n\left(M^\infty_{<\varepsilon}\right), \tag{5.3}$$

where

$$\tilde{\beta}_j^n(\varepsilon_1) = \frac{\pi}{\tilde{l}_j^n}\left(\pi - 2\arcsin\left(\frac{\sinh\left(\tilde{l}_j^n/2\right)}{\sinh\varepsilon_1}\right)\right)$$

and

$$\beta_i^n(\varepsilon_1) = \frac{\pi}{2l_i^n}\left(\pi - 2\arcsin\left(\frac{\sinh l_i^n}{\sinh\varepsilon_1}\right)\right).$$

Since $\frac{w_i^n}{l_i^n} \to 1$ and $\frac{\widetilde{w}_i^n}{\tilde{l}_i^n} \to 1$, there exists a number N_2 such that for every $n > N_2$ one has $\alpha_{j,\pm}^n < \tilde{\beta}_j^n(\varepsilon_1)$ and $\alpha_i^n < \beta_i^n(\varepsilon_1)$. Therefore, for all $n > N_2$ and for all i,j one obtains

$$\widetilde{\mathcal{C}}_j^n(\alpha_{j,-}^n,\alpha_{j,+}^n) \subset \widetilde{\mathcal{C}}_i^n(\tilde{\beta}_j^n(\varepsilon_1),\tilde{\beta}_j^n(\varepsilon_1)), \quad \mathcal{C}_i^n(0,\alpha_i^n) \subset \mathcal{C}_i^n(0,\beta_i^n(\varepsilon_1)) \tag{5.4}$$

Then (5.3) and (5.4) imply that for all $n > \max\{N_1,N_2\}$ one has

$$M \setminus (\Psi^n)^{-1}M_{<\varepsilon}^\infty \subseteq M \setminus \left(\cup_{i=1}^{\tilde{s}}\,\widetilde{\mathcal{C}}_i^n(\alpha_{i,-}^n,\alpha_{i,+}^n)\bigcup\cup_{i=1}^{s}\mathcal{C}_i^n(0,\alpha_i^n)\right) = \cup_{j=1}^r M_j^n(\alpha^n).$$

Applying Ψ^n we then get

$$M_{\geqslant\varepsilon}^\infty \subseteq \cup_{j=1}^r\check{\Omega}_j^n.$$

Since the domain $M_{\geqslant\varepsilon}^\infty$ exhausts M_∞ as ε goes to 0 we get the same for the domains $\check{\Omega}_j^n$ as n goes to ∞ and the claim follows. ☐

Applying the conformal transformations to (5.2) one has

$$\Lambda_k(M,c_n) \geqslant$$
$$\max\left(\sum_{i=1}^{s}\Lambda_{r_i}^N(\Omega_i^n,[g_{can}]) + \sum_{i=1}^{\tilde{s}}\Lambda_{\tilde{r}_i}^N(\widetilde{\Omega}_i^n,[g_{can}]) + \sum_{j=1}^{m+\widetilde{m}}\Lambda_{k_j}^N(\check{\Omega}_j^n,[(\Psi^n)^*h_n])\right). \tag{5.5}$$

By Corollary 4.7 one has that the first two terms on the right hand side converge to $\Lambda_{r_i}(\mathbb{RP}^n)$ and $\Lambda_{\widetilde{r}_i}(\mathbb{S}^n)$ respectively.

Lemma 5.2. *Let $\widehat{M_j^\infty} \subset \widehat{M_\infty}$ be a closure of M_j^∞. Then for all m one has*

$$\liminf_{n\to\infty}\Lambda_m^N(\check{\Omega}_j^n,[(\Psi^n)^*h_n]) \geqslant \Lambda_m(\widehat{M_j^\infty},[\widehat{h_\infty}]).$$

Proof. Fix $\varepsilon > 0$. An application of Corollary 4.7 to a compact exhaustion of M_j^∞ yields the existence of a compact $K \subset M_j^\infty \subset \widehat{M_j^\infty}$ such that

$$|\Lambda_m(\widehat{M_j^\infty},[\widehat{h_\infty}]) - \Lambda_m(K,[\widehat{h_\infty}])| < \varepsilon.$$

Since $\check{\Omega}_j^n$ exhaust M_j^∞, then for all large enough n one has $K \subset \check{\Omega}_j^n$. Then, by Proposition 4.5

$$\Lambda_m^N(\check{\Omega}_j^n,[(\Psi^n)^*h_n]) \geqslant \Lambda_m^N(K,[(\Psi^n)^*h_n]).$$

Taking lim inf of both sides in the above inequality and using Proposition 4.2 yields

$$\liminf_{n\to\infty}\Lambda_m^N(\check{\Omega}_j^n,[(\Psi^n)^*h_n]) \geqslant \Lambda_m^N(K,[\widehat{h_\infty}]) > \Lambda_m(\widehat{M_j^\infty},[\widehat{h_\infty}]) - \varepsilon.$$

Since ε is arbitrary, this completes the proof. $\square$

Finally, taking $\liminf_{n\to\infty}$ in (5.5) completes the proof of (5.1).

5.2. Inequality $\leqslant$.

We proceed with the inverse inequality,

$$\limsup_{n\to\infty} \Lambda_k(M,c_n) \leqslant$$
$$\max\Big(\sum_{i=1}^{\widetilde{m}} \Lambda_{\widetilde{k}_i}(\widetilde{\Sigma}_{\widetilde{\gamma}_i}, c_\infty) + \sum_{i=1}^{m} \Lambda_{k_i}(\Sigma_{\gamma_i}, c_\infty) + \sum_{i=1}^{\widetilde{s}} \Lambda_{\widetilde{r}_i}(\mathbb{S}^2) + \sum_{i=1}^{s} \Lambda_{r_i}(\mathbb{RP}^2)\Big), \tag{5.6}$$

In orientable case, this is essentially proved in [**98**, Section 7]. Below we outline the ideas of the proof and show the necessary modifications in the non-orientable case.

Let us choose a subsequence c_{n_m} such that

$$\lim_{n_m\to\infty} \Lambda_k(M,c_{n_m}) = \limsup_{n\to\infty} \Lambda_k(M,c_n).$$

We immediately relabel the subsequence and denote it by $\{c_n\}$. This way we can choose further subsequences without changing the value of $\limsup$.

Case 1. Suppose that up to a choice of a subsequence the following inequality holds

$$\Lambda_k(M,c_n) > \Lambda_{k-1}(M,c_n) + 8\pi.$$

Then by [**98**, Theorem 2] in the conformal class c_n there exists a unit volume metric g_n induced from a harmonic immersion Φ_n to some $N(n)$-dimensional sphere $\mathbb{S}^{N(n)}$, i.e.

$$g_n = \frac{|\nabla\Phi_n|^2_{h_n}}{\Lambda_k(M,c_n)} h_n$$

and such that $\lambda_k(g_n) = \Lambda_k(M,c_n)$. Here the metric h_n is the canonical representative in the conformal class c_n. It is known that for any compact surface the multiplicity of $\lambda_k(g_n)$ is bounded from above by a constant depending only on k and γ (see for instance [**10**] for orientable surfaces and [**5, 87**] for non-orientable surfaces). Therefore, one can choose the number $N(n)$ large enough such that $N(n)$ does not depend on n.

Assume that for the sequence $\{c_n\}$ the following inequality holds

$$\lim_{n\to\infty} \Lambda_k(M,c_n) >$$
$$\max\left(\sum_{i=1}^{\widetilde{m}} \Lambda_{\widetilde{k}_i}(\widetilde{\Sigma}_{\widetilde{\gamma}_i}, c_\infty) + \sum_{i=1}^{m} \Lambda_{k_i}(\Sigma_{\gamma_i}, c_\infty) + \sum_{i=1}^{\widetilde{s}} \Lambda_{\widetilde{r}_i}(\mathbb{S}^2) + \sum_{i=1}^{s} \Lambda_{r_i}(\mathbb{RP}^2)\right). \tag{5.7}$$

Proposition 5.3. *For $1 \leqslant i \leqslant s$ there exist integers $t_i \geqslant 0$, non-negative sequences $\{a^n_{i,l}\}, \{b^n_{i,l}\}$ with $1 \leqslant l \leqslant t_i$ and a sequence $\{\alpha^n_i\}$ such that*

$$0 \leqslant a^n_{i,t_i} \ll b^n_{i,t_i} \ll ... \ll a^n_{i,1} \ll b^n_{i,1} \ll a^n_{i,0} = \alpha^n_i \ll w^n_i$$

and

$$m_{i,l} = \lim_{n\to\infty} Vol(\mathcal{C}^n_i(a^n_{i,l},b^n_{i,l}), g_n) > 0.$$

Similarly for $1 \leqslant i \leqslant \tilde{s}$ there exist integers $\tilde{t}_i \geqslant 0$, sequences $\{a^n_{i,l}\}, \{b^n_{i,l}\}$ where $1 \leqslant l \leqslant \tilde{t}_i$ and sequences $\{\alpha^n_{i,\pm}\}$ such that

$$-w^n_i \ll \alpha^n_{i,-} = b^n_{i,0} \ll a^n_{i,1} \ll b^n_{i,1} \ll \ldots \ll a^n_{i,t_i} \ll b^n_{i,t_i+1} \ll a^n_{i,t_{i+1}} = \alpha^n_{i,+} \ll w^n_i$$

and

$$\widetilde{m}_{i,l} = \lim_{n\to\infty} Vol(\widetilde{\mathcal{C}}^n_i(a^n_{i,l},b^n_{i,l}), g_n) > 0.$$

Moreover, there exists a set $J \subset \{1, \ldots ,m + \widetilde{m}\}$ such that for every $j \in J$ one has

$$m_j = \lim_{n\to\infty} Vol(M^n_j(\alpha_n), g_n) > 0$$

satisfying

$$\sum_{i=1}^{s}\sum_{l=1}^{t_i} m_{i,l} + \sum_{i=1}^{\tilde{s}}\sum_{l=1}^{\tilde{t}_i} \widetilde{m}_{i,l} + \sum_{j\in J} m_j = 1,$$

with $\sum_{i=1}^{s} t_i + \sum_{i=1}^{\tilde{s}} \tilde{t}_i \leqslant k$ is maximal.

Proof. The proof follows the proofs of Claim 16, Claim 17 by [**98**]. Precisely, denying the proposition one can construct $k+1$ test-functions such that $\lambda_k(g_n) \leqslant o(1)$ which contradicts inequality (1.1). □

We proceed with considering a sequence $\{d^n_{i,l}\}$ where $1 \leqslant i \leqslant s$ and $1 \leqslant l \leqslant t_i$ such that

$$\lim_{n\to\infty} Vol(\mathcal{C}^n_i(a^n_{i,l},d^n_{i,l}),g_n) = \lim_{n\to\infty} Vol(\mathcal{C}^n_i(d^n_{i,l},b^n_{i,l}),g_n) = m_{i,l}/2$$

and a sequence $\{\widetilde{d}^n_{i,l}\}$ where $1 \leqslant i \leqslant \tilde{s}$ and $1 \leqslant l \leqslant \tilde{t}_i$ such that

$$\lim_{n\to\infty} Vol(\widetilde{\mathcal{C}}^n_i(a^n_{i,l},\widetilde{d}^n_{i,l}),g_n) = \lim_{n\to\infty} Vol(\widetilde{\mathcal{C}}^n_i(\widetilde{d}^n_{i,l},b^n_{i,l}),g_n) = \widetilde{m}_{i,l}/2.$$

For $1 \leqslant i \leqslant \tilde{s}$ let $\widetilde{q}^n_{i,l} \ll a^n_{i,l}$, $\widetilde{q}^n_{i,l} \to +\infty$. Consider the conformal maps $\widetilde{\Psi}^n_{i,l}\colon \left(\widetilde{\mathcal{C}}^n_i(a^n_{i,l} - \widetilde{q}^n_{i,l},b^n_{i,l} + \widetilde{q}^n_{i,l}),c_n\right) \to (\mathbb{S}^2,[g_{can}])$ defined as

$$\widetilde{\Psi}^n_{i,l}(t,\theta) = \frac{1}{e^{2(t-\widetilde{d}^n_{i,l})}+1}(2e^{t-\widetilde{d}^n_{i,l}}\cos\theta, 2e^{t-\widetilde{d}^n_{i,l}}\sin\theta, e^{2(t-\widetilde{d}^n_{i,l})}-1).$$

Let $\widetilde{\Omega}^n_{i,l} \subset \mathbb{S}^2$ be the image of this map. Let $\widetilde{\Phi}^n_{i,l} = \Phi_n \circ (\widetilde{\Psi}^n_{i,l})^{-1} : (\widetilde{\Omega}^n_{i,l}, g_{can}) \to (\mathbb{S}^N, g_{can})$. Then $\widetilde{\Phi}^n_{i,l}$ is harmonic since Φ_n is harmonic and $\widetilde{\Psi}^n_{i,l}$ is conformal. Moreover, it is shown in [**98**] that the measure $\mathbf{1}_{\widetilde{\Omega}^n_{i,l}}|\nabla\widetilde{\Phi}^n_{i,l}|^2_{g_{can}} dv_{g_{can}}$ does not concentrate at the poles (0,0,1) and

$(0,0,-1)$ of $\mathbb{S}^2$. Indeed, if the measure concentrated at the poles then one would obtain a contradiction with the maximality of $\sum_{i=1}^{s} t_i + \sum_{i=1}^{\tilde{s}} \tilde{t}_i$.

Similarly, for $1 \leqslant i \leqslant s$ if $a_{i,l}^n \neq 0$ let $0 < q_{i,l}^n \ll a_{i,l}^n$, $q_{i,l}^n \to +\infty$, otherwise let $0 < q_{i,l}^n \ll b_{i,l}^n$, $q_{i,l}^n \to +\infty$. If $a_{i,l}^n \neq 0$ consider the conformal maps $\Psi_{i,l}^n \colon \left(\mathcal{C}_i^n(a_{i,l}^n - q_{i,l}^n, b_{i,l}^n + q_{i,l}^n), c_n\right) \to (\mathbb{S}^2,[g_{can}])$ defined on the orientable double covers as

$$\Psi_{i,l}^n(t,\theta) = \frac{1}{e^{2(t-d_{i,l}^n)}+1}(2e^{t-d_{i,l}^n}\cos\theta, 2e^{t-d_{i,l}^n}\sin\theta, e^{2(t-d_{i,l}^n)}-1).$$

If $a_{i,l}^n = 0$, then $\Psi_{i,l}^n$ is defined in the same way, the only difference is that the domain is $\mathcal{C}_i^n(a_{i,l}^n, b_{i,l}^n + q_{i,l}^n)$. Either way, let $\Omega_{i,l}^n \subset \mathbb{RP}^2$ be the image of this map. Let $\Phi_{i,l}^n = \Phi_n \circ (\Psi_{i,l}^n)^{-1} : (\Omega_{i,l}^n, g_{can}) \to (\mathbb{S}^N, g_{can})$. Then $\widetilde{\Phi}_{i,l}^n$ is harmonic since Φ_n is harmonic and $\Psi_{i,l}^n$ is conformal. Similarly to the previous paragraph one has that the measure $\mathbf{1}_{\Omega_{i,l}^n}|\nabla\Phi_{i,l}^n|_{g_{can}}^2 dv_{g_{can}}$ does not concentrate at the antipodal image of the pole $(0,0,1)$ in $\mathbb{RP}^2$.

The exactly same procedure can be carried out for components $M_j^n(\alpha)$, $j \in J$. The only difference is that now we use the restriction of diffeomorphisms Ψ^n given by Proposition 2.6 instead of the explicit harmonic map as above. As a result, one obtains domains $\check{\Omega}_j^n \subset M_\infty$ and harmonic maps $\check{\Phi}_j^n \colon \check{\Omega}_j^n \to \mathbb{S}^N$ such that the measure $\mathbf{1}_{\check{\Omega}_j^n}|\nabla\check{\Phi}_j^n|_{g_{can}}^2 dv_{g_{can}}$ does not concentrate at the marked points of $\widehat{M_\infty}$.

As the next step, one applies bubble convergence theorem for harmonic maps and the non-concentration results above to choose a subsequence such that the measures $\mathbf{1}_{\widetilde{\Omega}_{i,l}^n}|\nabla\widetilde{\Phi}_{i,l}^n|_{g_{can}}^2 dv_{g_{can}}$, $\mathbf{1}_{\Omega_{i,l}^n}|\nabla\Phi_{i,l}^n|_{g_{can}}^2 dv_{g_{can}}$ and $\mathbf{1}_{\check{\Omega}_j^n}|\nabla\check{\Phi}_j^n|_{g_{can}}^2 dv_{g_{can}}$ converge in $*$-weak topology. One then uses eigenfunctions of limiting measures (and eigenfunctions on bubbles of $\{\Psi_j^n\}$ if bubbles exist) as test-functions for $\lambda_k(M,g_n)$. Since bubble convergence does not require the domain to be orientable and the construction of eigenfunctions supported on bubbles is local, this argument carries over to the non-orientable case without any changes. For further details, see [**98**, Section 7].

As a result, one obtains the following inequality

$$\limsup_{n\to\infty} \Lambda_k(M,c_n) \leqslant$$
$$\sum_{j\in J}\Lambda_{k_j}(\widehat{M_j^\infty},c_\infty) + \sum_{i=1}^{\widetilde{s}}\sum_{l=1}^{\widetilde{t}_i}\Lambda_{\widetilde{r}_{i,l}}(\mathbb{S}^2) + \sum_{i=1}^{s}\left(\sum_{l=1}^{t_i-1}\Lambda_{r_{i,l}}(\mathbb{S}^2) + \Lambda_{r_{i,t_i}}(S_i)\right),$$

where $S_i = \mathbb{RP}^2$ if the sequence $\{a_{i,t_i}^n\}_n$ contains infinitely many zeros, $S_i = \mathbb{S}^2$ otherwise, and

$$\sum_{j\in J} k_j + \sum_{i=1}^{\widetilde{s}}\sum_{l=1}^{\widetilde{t}_i}\widetilde{r}_{i,l} + \sum_{i=1}^{s}\sum_{l=1}^{t_i} r_{i,l} \leqslant k.$$

Finally, an application of inequality (1.2) allows us to group together the terms with the same index i to obtain inequality (5.6).

Case 2. Assume that up to a choice of a subsequence the following inequality holds

$$\Lambda_k(M,c_n) \leqslant \Lambda_{k-1}(M,c_n) + 8\pi$$

then we prove inequality (5.6) by induction.

Note that if $k=1$ then by Theorem 1.3 $\Lambda_1(M,[h_n]) > 8\pi$, i.e. $k=1$ falls under Case 1. Therefore, the inequality (5.6) holds for $k=1$. This is the base of induction.

Suppose that the proposition holds for all numbers $k' \leqslant k$. We show that it also holds for $k+1$. Indeed, one has

$$\Lambda_{k+1}(M,c_n) \leqslant \Lambda_k(M,c_n) + 8\pi = \Lambda_k(M,c_n) + \Lambda_1(\mathbb{S}^2)$$

and inequality (5.6) holds then we get

$$\begin{aligned}
&\lim_{n\to\infty} \Lambda_{k+1}(M,c_n) \leqslant \\
&\max\Big(\sum_{i=1}^{\widetilde{m}} \Lambda_{\widetilde{k}_i}(\widetilde{\Sigma}_{\widetilde{\gamma}_i}, c_\infty) + \sum_{i=1}^{m} \Lambda_{k_i}(\Sigma_{\gamma_i}, c_\infty) + \sum_{i=1}^{\widetilde{s}} \Lambda_{\widetilde{r}_i}(\mathbb{S}^2) + \sum_{i=1}^{s} \Lambda_{r_i}(\mathbb{RP}^2)\Big) + \Lambda_1(\mathbb{S}^2) \leqslant \\
&\leqslant \max\Big(\sum_{i=1}^{\widetilde{m}} \Lambda_{\widetilde{k}'_i}(\widetilde{\Sigma}_{\widetilde{\gamma}_i}, c_\infty) + \sum_{i=1}^{m} \Lambda_{k'_i}(\Sigma_{\gamma_i}, c_\infty) + \sum_{i=1}^{\widetilde{s}} \Lambda_{\widetilde{r}'_i}(\mathbb{S}^2) + \sum_{i=1}^{s} \Lambda_{r'_i}(\mathbb{RP}^2)\Big),
\end{aligned}$$

where the term $\Lambda_1(\mathbb{S}^2)$ was absorbed by one of the terms inside max using inequality (1.2), and the last maximum is taken over all possible combinations of indices such that

$$\sum_{i=1}^{m} k'_i + \sum_{i=1}^{\widetilde{m}} \widetilde{k}'_i + \sum_{i=1}^{s} r'_i + \sum_{i=1}^{\widetilde{s}} \widetilde{r}'_i = k+1.$$

5.3. Non-hyperbolic case.

If $M = \mathbb{KL}$ or $M = \mathbb{T}^2$ the proof is very similar. Indeed, as it follows from the discussion in Section 2.5 for degenerating sequence one can find a collapsing geodesic and the whole surface becomes a flat collar of width $w_n \to +\infty$. An analog of Proposition 5.3 is proved in exactly the same way. The only difference in the rest of the proof is the fact that there is at most one domain $M^n_j(\alpha^n)$ and it is a flat cylinder or a Möbius band. Therefore, to construct $\check{\Phi}^n_j$ instead of the Deligne-Mumford compactification one uses the same construction as for $\widetilde{\Phi}^n_{i,l}$ or $\Phi^n_{i,l}$.

Third Chapter.

Degenerating sequences of conformal classes and the conformal Steklov spectrum

by

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Abstract. Let Σ be a compact surface with boundary. For a given conformal class c on Σ the functional $\sigma_k^*(\Sigma,c)$ is defined as the supremum of the $k-$th normalized Steklov eigenvalue over all metrics on c. We consider the behaviour of this functional on the moduli space of conformal classes on Σ. A precise formula for the limit of $\sigma_k^*(\Sigma,c_n)$ when the sequence $\{c_n\}$ degenerates is obtained. We apply this formula to the study of natural analogs of the Friedlander-Nadirashvili invariants of closed manifolds defined as $\inf_c \sigma_k^*(\Sigma,c)$, where the infimum is taken over all conformal classes c on Σ. We show that these quantities are equal to $2\pi k$ for any surface with boundary. As an application of our techniques we obtain new estimates on the $k-$th normalized Steklov eigenvalue of a non-orientable surface in terms of its genus and the number of boundary components.
Keywords: conformal Steklov spectrum, moduli space of conformal classes, upper bounds

1. Introduction and main results

Let (Σ,g) be a compact Riemannian surface with boundary. In this paper we always assume that Σ is connected and the boundary of Σ is non-empty and smooth. Consider *the Steklov problem* defined in the following way

$$\begin{cases} \Delta u = 0 & \text{in } \Sigma, \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Sigma, \end{cases}$$

where $\Delta = -div_g \circ grad_g$ is the Laplace-Beltrami operator and $\frac{\partial}{\partial n}$ is the outward unit normal vector field along the boundary. The collection of all numbers σ for which the Steklov problem admits a solution is called the *Steklov spectrum* of the surface Σ. The Steklov spectrum is a discrete set of real numbers called Steklov eigenvalues with finite multiplicities satisfying the following condition (see e.g. [**41**])

$$0 = \sigma_0(g) < \sigma_1(g) \leqslant \sigma_2(g) \leqslant ... \nearrow +\infty.$$

The Steklov spectrum enables us to define the following homothety-invariant functional on the set $\mathcal{R}(\Sigma)$ of Riemannian metrics on Σ

$$\overline{\sigma}_k(\Sigma,g) := \sigma_k(g)L_g(\partial\Sigma),$$

where $L_g(\partial\Sigma)$ stands for the length of the boundary of Σ in the metric g. The functional $\overline{\sigma}_k(\Sigma,g)$ is called the *$k-th$ normalized Steklov eigenvalue*. It was shown in [**16**] (see also [**43, 63**]) that if Σ is an orientable surface then the functional $\overline{\sigma}_k(\Sigma,g)$ is bounded from above. Moreover, the following theorem holds

Theorem 1.1 ([**39**]). *Let (Σ,g) be a compact orientable surface of genus γ with l boundary components. Then one has*

$$\overline{\sigma}_k(\Sigma,g) \leqslant 2\pi k(\gamma + l).$$

In this paper we prove that a similar estimate holds for non-orientable surfaces.

Theorem 1.2. *Let Σ be a compact non-orientable surface of genus γ with l boundary components. Then one has*

$$\overline{\sigma}_k(\Sigma,g) \leqslant 4\pi k(\gamma + 2l).$$

Here the genus of a non-orientable surface is defined as the genus of its orientable cover.

Remark 1.3. *The estimate in Theorem 1.1 has been improved in* [**52**] *by a bound which is linear in $k+\gamma+l$ instead of $k(\gamma+l)$. However, the proof of this result uses orientability in an essential way, see* [**52**, Section 6]. *It would be interesting to obtain a similar improvement in Theorem 1.2.*

Theorems 1.1 and 1.2 enable us to define the following functionals

$$\sigma_k^*(\Sigma) := \sup_{\mathcal{R}(\Sigma)} \overline{\sigma}_k(\Sigma,g),$$

and

$$\sigma_k^*(\Sigma,[g]) := \sup_{[g]} \overline{\sigma}_k(\Sigma,g).$$

Remark 1.4. *Note that we cannot define the functionals $\sigma_k^*(\Sigma)$ and $\sigma_k^*(\Sigma,[g])$ in higher dimensions. Indeed, it was proved in the paper* [**17**] *that if* $\dim(M) \geqslant 3$ *then the functional* $\overline{\sigma}_k(M,g)$ *is not bounded from above on the set of Riemannian metrics* $\mathcal{R}(M)$*. Moreover, it is not even bounded from above in the conformal class* $[g]$*.*

The functional $\sigma_k^*(\Sigma)$ is an object of intensive research during the last decade (see e.g. [**30, 33, 15, 99, 38, 71**]).

The functional $\sigma_k^*(\Sigma,[g])$ which is called the *$k-th$ conformal Steklov eigenvalue* is less studied. Let us mention some results concerning $\sigma_k^*(\Sigma,[g])$. First since the disc admits the unique conformal structure one can conclude that $\sigma_k^*(\mathbb{D}^2,[g_{can}]) = \sigma_k^*(\mathbb{D}^2)$, where g_{can} stands for the Euclidean metric on $\mathbb{D}^2$ with unit boundary length. The value of $\sigma_k^*(\mathbb{D}^2)$ is known: $\sigma_k^*(\mathbb{D}^2) = 2\pi k$ (see [**107**] for $k=1$ and [**40**] for all $k \geqslant 1$). The functional $\sigma_k^*(\Sigma,[g])$ is the main research object of the paper [**99**].

Theorem 1.5 ([**99**])**.** *For every Riemannian metric g on a compact surface* Σ *with boundary one has*

$$\sigma_k^*(\Sigma,[g]) \geqslant \sigma_{k-1}^*(\Sigma,[g]) + \sigma_1^*(\mathbb{D}^2, [g_{can}]), \tag{1.1}$$

particularly

$$\sigma_k^*(\Sigma,[g]) \geqslant 2\pi k. \tag{1.2}$$

Moreover, if the inequality 1.1 is strict then there exists a Riemannian metric $\tilde{g} \in [g]$ *such that* $\overline{\sigma}_k(\Sigma,\tilde{g}) = \sigma_k^*(\Sigma,[g])$*.*

New interesting results about the functional $\sigma_k^*(\Sigma,[g])$ were recently obtained in the paper [**58**].

Remark 1.6. *The result analogous to Theorem 1.5 for the conformal spectrum of the Laplace-Beltrami operator on closed surfaces also holds (see* [**84, 85, 96, 98, 57**]*). For further information concerning the spectrum of the Laplace-Beltrami operator on closed surfaces see the surveys* [**94, 95**] *and references therein.*

It is easy to see that the connection between the functionals $\sigma_k^*(\Sigma)$ and $\sigma_k^*(\Sigma,[g])$ is expressed by the formula

$$\sigma_k^*(\Sigma) = \sup_{[g]} \sigma_k^*(\Sigma,[g]).$$

One can ask what do we get if we replace $\sup_{[g]}$ by $\inf_{[g]}$ in this formula? In this case we get the following quantity

$$I_k^\sigma(\Sigma) := \inf_{[g]} \sigma_k^*(\Sigma,[g]),$$

It is an analog of the Friedlander-Nadirashvili invariant of closed manifolds. The first Friedlander-Nadirashvili invariant of a closed manifold was introduced in the paper [**34**] in 1999. The $k-$th Nadirashvili-Friedlander invariant of a closed surface has been recently studied in the paper [**55**].

In the study of functionals like $\sigma_k^*(\Sigma)$ and $I_k^\sigma(\Sigma)$ one considers maximizing and minimizing sequences of conformal classes $\{c_n\}$ on the *moduli space of conformal classes on* Σ, i.e. $\sigma_k^*(\Sigma,c_n) \to \sigma_k^*(\Sigma)$ or $\sigma_k^*(\Sigma,c_n) \to I_k^\sigma(\Sigma)$ as $n \to \infty$. Due to the Uniformization theorem conformal classes on Σ are in one-to-one correspondence (up to an isometry) with metrics on Σ of constant Gauss curvature and geodesic boundary. Therefore, any sequence of conformal classes $\{c_n\}$ on Σ corresponds to a sequence of Riemannian surfaces of constant Gauss curvature and geodesic boundary $\{(\Sigma,h_n)\}$, $h_n \in c_n$ and we can consider the moduli space of conformal classes on Σ as the set of all (Σ,h), where h is a metric of constant Gauss curvature and geodesic boundary, endowed with $C^\infty-$topology (see Section 4). Note that the moduli space of conformal structures is a non-compact topological space. For any sequence $\{c_n\}$ there are two possible scenarios: either this sequence remains in a compact part of the moduli space or it escapes to infinity. Let $(\Sigma_\infty, c_\infty)$ denote the *limiting space*, i.e. $(\Sigma_\infty, c_\infty) = \lim_{n\to\infty}(\Sigma,c_n)$. We compactify Σ_∞ if necessary. Let $\widehat{\Sigma_\infty}$ denote the compactified limiting space. It turns out that if the first scenario realizes then we get $\widehat{\Sigma_\infty} = \Sigma$ and c_∞ is a genuine conformal class on Σ for which the value $\sigma_k^*(\Sigma)$ or $I_k^\sigma(\Sigma)$ is attained. If the second scenario realizes then we say that the sequence $\{c_n\}$ *degenerates*. It turns out that in this case there exists a finite collection of pairwise disjoint geodesics for the metrics h_n whose lengths in h_n tend to 0 as n tends to ∞. We refer to these geodesics as *pinching* or *collapsing*. They can be of the following three types: the collapsing boundary components, the collapsing geodesics with no self-intersection having two points of intersection with $\partial\Sigma$ and the collapsing geodesics with no self-intersection and which do not cross $\partial\Sigma$. Note that in this case the topology of Σ necessarily changes when we pass to the limit as $n \to \infty$, i.e. the compact surfaces $\widehat{\Sigma}_\infty$ and Σ belong to different topological types. In particular, the surface $\widehat{\Sigma}_\infty$ can be disconnected (see Figure 4). We refer to Section 4 for more details.

The following theorem establishes the correspondence between $\sigma_k^*(\widehat{\Sigma}_\infty,c_\infty)$ and the limit of $\sigma_k^*(\Sigma,c_n)$ when the sequence of conformal classes c_n degenerates. It is an analog of [**55**, Theorem 2.8] for the Steklov setting.

Theorem 1.7. *Let Σ be a compact surface of genus γ with $l > 0$ boundary components and let $c_n \to c_\infty$ be a degenerating sequence of conformal classes. Consider the corresponding sequence $\{h_n\}$ of metrics of constant Gauss curvature and geodesic boundary. Suppose*

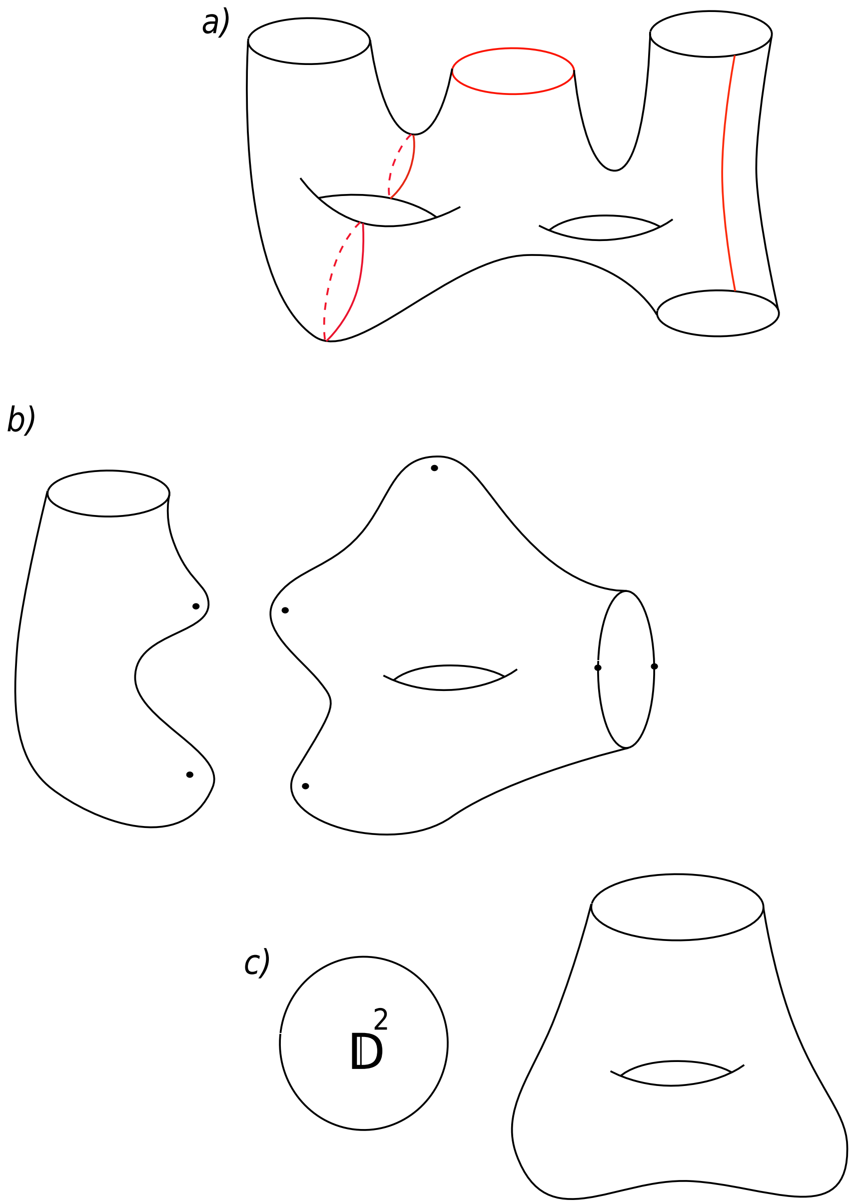

Fig. 4. An example of a degenerating sequence of conformal classes $\{c_n\}$ on a surface Σ of genus 2 with 4 boundary components. a) The *red* curves correspond to collapsing geodesics for the sequence of metrics of constant Gauss curvature and geodesic boundary $\{h_n\}$, $h_n \in c_n$ corresponding to the degenerating sequence of conformal classes $\{c_n\}$. b) The compactified limiting space $\widehat{\Sigma_\infty}$. The black points correspond to the points of compactification. c) The surface $\widehat{\Sigma_\infty}$ is homeomorphic to the disjoint union of a disc and a surface of genus 1 with 1 boundary component.

that there exist s_1 collapsing boundary components and s_2 collapsing geodesics with no self-intersection which cross the boundary at two points. Moreover, suppose that $\widehat{\Sigma_\infty}$ has m connected components Σ_{γ_i,l_i} of genus γ_i with $l_i > 0$ boundary components, $\gamma_i + l_i < \gamma + l$, $i = 1, \dots, m$. Then one has

$$\lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = \max\Big(\sum_{i=1}^{m} \sigma_{k_i}^*(\Sigma_{\gamma_i,l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2)\Big), \tag{1.3}$$

where the maximum is taken over all possible combinations of indices such that

$$\sum_{i=1}^{m} k_i + \sum_{i=1}^{s_1+s_2} r_i = k.$$

Remark 1.8. *Let Σ denote either cylinder or the Möbius band. Theorem 1.7 particularly implies that if the sequence of conformal classes $\{c_n\}$ on Σ degenerates then we necessarily have:*

$$\lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = 2\pi k.$$

Remark 1.9. *In Theorem 1.7 the sequence $\{h_n\}$ can also have collapsing geodesics not crossing the boundary of Σ. Moreover, it can happen that the limiting space $\widehat{\Sigma_\infty}$ has closed components (see Figure 5). Anyway, in Theorem 1.7 we take only components of $\widehat{\Sigma_\infty}$ which have non-empty boundary.*

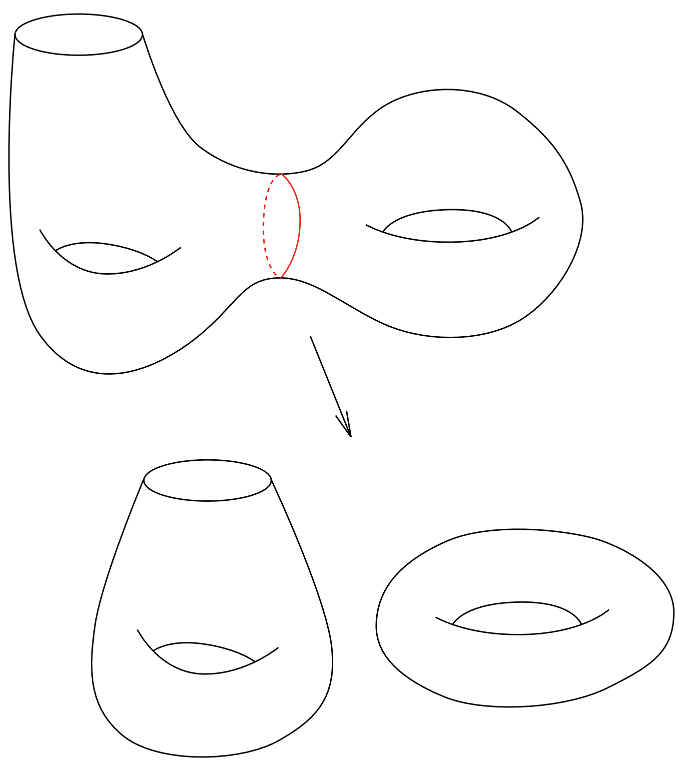

Fig. 5. An example of a degenerating sequence of conformal classes $\{c_n\}$ on a surface of genus 2 with 1 boundary components such that the limiting space contains a closed component. In Theorem 1.7 we take only the component on the left which has non-empty boundary. Note that in this case $s_1 = s_2 = 0$.

The main tool that we use in the proof of Theorem 1.7 is the *Steklov-Neumann boundary problem* also known as the *sloshing problem*. Let Ω be a Lipschitz domain in (Σ, g) such that

$\overline{\Omega} \cap \partial\Sigma = \partial^S\Omega \neq \varnothing$. Let $\partial^N\Omega = \partial\Omega \setminus \partial\Sigma$. Then the Steklov-Neumann problem is defined as:

$$\begin{cases} \Delta_g u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial^N\Omega, \\ \frac{\partial u}{\partial n} = \sigma^N u & \text{on } \partial^S\Omega. \end{cases} \tag{1.4}$$

The numbers σ^N for which the Steklov-Neumann problem admits a solution are called *Steklov-Neumann eigenvalues*. It is known (see [**4**] and references therein) that the set of Steklov-Neumann eigenvalues is not empty and discrete

$$0 = \sigma_0^N(g) < \sigma_1^N(g) \leqslant \sigma_2^N(g) \leqslant \ldots \nearrow +\infty.$$

Every Steklov-Neumann eigenvalue admits the following variational characterization:

$$\sigma_k^N(g) = \inf_{V_k \subset \mathcal{H}^1(\Omega)} \sup_{0 \neq u \in V_k} \frac{\int_\Omega |\nabla u|^2 dv_g}{\int_{\partial^S\Omega} u^2 ds_g}, \tag{1.5}$$

where the infimum is taken over all $k-$dimensional subspaces of the space $\mathcal{H}^1(\Omega) = \{u \in H^1(\Omega,g) \mid \int_{\partial^S\Omega} u ds_g = 0\}$.

Similarly to the case of the Steklov problem we define normalized Steklov-Neumann eigenvalues as

$$\overline{\sigma}_k^N(\Omega, \partial^S\Omega, g) := \sigma_k^N(g) L_g(\partial^S\Omega).$$

In this notation we always indicate the Steklov part of the boundary at the second place. Sometimes we also use the notation $\sigma_k^N(\Omega, \partial^S\Omega, g)$ for $\sigma_k^N(\Omega, g)$ to emphasize that the Steklov boundary condition is imposed on $\partial^S\Omega$.

Remark 1.10. *Consider Ω as a surface with Lipschitz boundary. It also follows from* [**63**, Theorem A_k] *that the quantity $\overline{\sigma}_k^N(\Omega, \partial^S\Omega, g)$ is bounded from above on $[g]$ and we can define the invariant $\sigma_k^{N*}(\Omega, \partial^S\Omega, [g])$ in the same way as the invariant $\sigma_k^*(\Sigma, [g])$.*

Theorem 1.7 enables us to establish the value of I_k^σ.

Theorem 1.11. *Let Σ be a compact surface with boundary. Then one has $I_k^\sigma(\Sigma) = I_k^\sigma(\mathbb{D}^2) = 2\pi k$.*

1.1. Discussion

Let us discuss the estimate obtained in Theorem 1.2. The first estimate on $\overline{\sigma}_1(\Sigma, g)$, where Σ is a non-orientable surface of genus γ with boundary follows from the papers [**63, 51**]:

$$\overline{\sigma}_1(\Sigma, g) \leqslant 16\pi \left[\frac{\gamma + 3}{2}\right], \tag{1.6}$$

were $[x]$ stands for the integer part of the number x.

Very recently in the paper [**58**] estimate (1.6) has been improved and extended for $k = 2$: consider Σ as a domain with smooth boundary on a closed surface M, then one has

$$\overline{\sigma}_k(\Sigma,g) \leqslant \Lambda_k(M),\ k = 1,2. \tag{1.7}$$

In this estimate $\Lambda_k(M) := \sup_{g\in\mathcal{R}(M)} \lambda_k(g)Vol(M,g)$, where $\lambda_k(g)$ is the $k-$th Laplace eigenvalue of the metric g, $Vol(M,g)$ is the volume of M is the metric g and $\mathcal{R}(M)$ is the set of Riemannian metrics on M. Note that estimate (1.7) does not depend on the number of boundary components. Combining estimate (1.7) with our estimate we get

$$\overline{\sigma}_k(\Sigma,g) \leqslant \min\{\Lambda_k(M), 4\pi k(\gamma + 2l)\},\ k = 1,2.$$

Particularly, for the Möbius band one has

$$\overline{\sigma}_k(\mathbb{MB},g) \leqslant \min\{\Lambda_k(\mathbb{RP}^2), 8\pi k\},\ k = 1,2,$$

since $\mathbb{MB} \subset \mathbb{RP}^2$. The value $\Lambda_k(\mathbb{RP}^2)$ is known for all k (see [**53**]): $\Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$. Hence

$$\overline{\sigma}_k(\mathbb{MB},g) \leqslant \min\{4\pi(2k+1), 8\pi k\} = 8\pi k,\ k = 1,2.$$

In the paper [**33**] it was shown that $\overline{\sigma}_1(\mathbb{MB},g) \leqslant 2\pi\sqrt{3}$ which is obviously $\leqslant 8\pi$.

We proceed with the discussion of the functional I^σ_k. Unlike Theorem 1.4 in [**55**] Theorem 1.11 says nothing about conformal classes on which the value $I^\sigma_k(\Sigma)$ is attained. We conjecture that

Conjecture 1.12. *The infimum $I^\sigma_k(\Sigma)$ is attained if and only if Σ is diffeomorphic to the disc $\mathbb{D}^2$.*

Note that this conjecture would be a corollary of the following one

Conjecture 1.13. *Let Σ be a compact surface non-diffeomorphic to the disc. Then for every conformal class c on Σ one has*

$$\sigma^*_1(\Sigma,c) > \sigma^*_1(\mathbb{D}^2) = 2\pi.$$

This conjecture is an analog of the Petrides rigidity theorem for the first conformal Laplace eigenvalue [**96**, Theorem 1]. Recently this conjecture was confirmed in the case of the cylinder and the Möbius band (see [**72**]). We plan to tackle Conjectures 1.12 and 1.13 in the subsequent papers.

Let us discuss the analogy between the quantity I^σ_k and the Friedlander-Nadirashvili invariant of closed surfaces I_k. In the paper [**55**] it was conjectured that I_k are invariants of cobordisms of closed surfaces (see Conjecture 1.8). Similarly, one can see that I^σ_k are invariants of cobordisms of compact surfaces with boundary. Let us recall that two compact surfaces with boundary $(\Sigma_1,\partial\Sigma_1)$ and $(\Sigma_2,\partial\Sigma_2)$ are called cobordant if there exists a 3-dimensional *manifold with corners* Ω whose boundary is $\Sigma_1 \cup_{\partial\Sigma_1} W \cup_{\partial\Sigma_2} \Sigma_2$, where W is a cobordism of $\partial\Sigma_1$ and $\partial\Sigma_2$ (i.e. W is a surface with boundary $\partial\Sigma_1 \sqcup \partial\Sigma_2$). Following [**7**] we denote a cobordism of two surfaces $(\Sigma_1,\partial\Sigma_1)$ and $(\Sigma_2,\partial\Sigma_2)$ by $(\Omega;\Sigma_1,\Sigma_2,W;\partial\Sigma_1,\partial\Sigma_2)$. One can easily see that the cobordisms of surfaces with boundary are trivial. Indeed, we can construct the following cobordism of a surface $(\Sigma,\partial\Sigma)$ and $(\varnothing,\varnothing)$: $(\Sigma\times[0,1];\Sigma\times\{0\},\varnothing,\partial\Sigma\times[0,1]\cup\Sigma\times\{1\};\partial\Sigma,\varnothing)$. A fundamental fact about cobordisms of surfaces with boundary is *Theorem about splitting cobordisms* (see [**7**, Theorem 4.18]) which says that every cobordism of compact surfaces with boundary can be split into a sequence of cobordisms given by a handle attachment and cobordisms given by a *half-handle* attachment. We refer to [**7**] for definitions and further information about cobordisms of compact manifolds with boundary. Analysing the proof of Theorem 1.11 one can remark that the value of I_k^σ does not change under handle and half-handle attachments. Since by this procedure any surface Σ can be reduced to the disc, we get $I_k^\sigma(\Sigma)=I_k^\sigma(\mathbb{D}^2)=2\pi k$.

Plan of the paper.

The paper is organized in the following way. In Section 2 we collect all the analytic facts which are necessary for the proof of Theorem 1.7. The main result here is Proposition 2.6. In Section 3 we prove Theorem 1.2 using the techniques developed in the previous section. Section 4 represents the geometric part of the paper. Here we describe convergence on the moduli space of conformal structures on a surface with boundary. Section 5 is devoted to the proof of Theorem 1.7. In Section 6 we deduce Theorem 1.11 from Theorem 1.7. Finally, Section 7 contains some auxiliary technical results.

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2. Analytic background

Here we provide a necessary analytic background that we will use in the proof of Theorem 1.7 in Section 5. The propositions in this section are analogs of the propositions in [**55**, Section 4]. We postpone the proof of a proposition to Section 7.2 every time when it follows the exactly same way as the proof of an analogous proposition in [**55**, Section 4].

2.1. Convergence of Steklov-Neumann spectrum

We start with the following convergence result.

Lemma 2.1. *Let (M, g) be a compact Riemannian manifold with boundary. Consider a finite collection $\{B_\epsilon(p_i)\}_{i=1}^l$ of geodesic balls of radius ϵ centred at some points $p_1, \ldots, p_l \in M$. Then the spectrum of the Steklov-Neumann problem*

$$\begin{cases} \Delta_g u = 0 & \text{in } M \setminus \cup_{i=1}^l B_\epsilon(p_i), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \cup_{i=1}^l \partial B_\epsilon(p_i) \setminus \partial M, \\ \frac{\partial u}{\partial n} = \lambda_k^N(M \setminus \cup_{i=1}^l B_\epsilon(p_i), g) u & \text{on } \partial M \setminus \cup_{i=1}^l \partial B_\epsilon(p_i) \end{cases} \tag{2.1}$$

converges to the Steklov spectrum of (M,g) as $\epsilon \to 0$.

Proof. For the sake of simplicity we only consider the case of one ball that we denote by B_ϵ centred at $p \in M$. First we consider the case when $B_\epsilon \cap \partial M \neq \varnothing$, i.e. $p \in \partial M$.

Let $\mathcal{E}(u)$ denote the extension of the function u by the unique solution of the problem

$$\begin{cases} \Delta_g \mathcal{E}(u) = 0 & \text{in } B_\epsilon, \\ \frac{\partial \mathcal{E}(u)}{\partial n} = 0 & \text{on } \partial M \cap \partial B_\epsilon, \\ \mathcal{E}(u) = u & \text{on } \partial B_\epsilon \setminus \partial M. \end{cases}$$

Claim 1. The operator $\mathcal{E}(u)$ is uniformly bounded.

Proof. The proof is similar to the proof of uniform boundedness of the harmonic continuation operator into small geodes balls [**100**, Example 1]. Fix $0 < r < \epsilon$ and let B_r denote a geodesic ball of radius r with the same center as B_ϵ. One has

$$||\mathcal{E}(u)||^2_{L^2(B_r,g)} \leqslant C||u||^2_{L^2(M\setminus B_r,g)} + C||\nabla u||^2_{L^2(M\setminus B_r,g)} \tag{2.2}$$

and

$$||\nabla \mathcal{E}(u)||^2_{L^2(B_r,g)} \leqslant C||\nabla u||^2_{L^2(M\setminus B_r,g)}. \tag{2.3}$$

Inequality (2.2) follows from estimate (7.1) and the trace inequality

$$||\mathcal{E}(u)||^2_{L^2(B_r,g)} \leqslant ||\mathcal{E}(u)||^2_{H^1(B_r,g)} \leqslant C||u||^2_{H^{1/2}(\partial B_r\setminus\partial M,g)} \leqslant C||u||^2_{H^1(M\setminus B_r,g)}.$$

Suppose that inequality (2.3) was false. Then there exists a sequence of functions $\{u_n\}$ in $H^1(M \setminus B_r,g)$ such that

$$||\nabla u_n||_{L^2(M\setminus B_r,g)} \leqslant 1/n$$

and

$$||\mathcal{E}(u_n)||_{L^2(B_r,g)} \geqslant 1.$$

Consider $\alpha_n = \frac{1}{Vol(M\setminus B_r,g)} \int_{M\setminus B_r} u_n dv_g$. We show that

$$||u_n - \alpha_n||_{H^1(M\setminus B_r,g)} \leqslant C/n.$$

Indeed, by the generalized Poincaré inequality one has

$$||u_n - \alpha_n||_{L^2(M\setminus B_r,g)} \leqslant C||\nabla u_n||_{L^2(M\setminus B_r,g)} \leqslant C/n$$

moreover

$$||\nabla(u_n - \alpha_n)||_{L^2(M\setminus B_r,g)} = ||\nabla u_n||_{L^2(M\setminus B_r,g)} \leqslant 1/n.$$

Note that $\mathcal{E}(u_n - \alpha_n) = \mathcal{E}(u_n) - \alpha_n$. Then we can prove inequality (2.3)

$$||\nabla\mathcal{E}(u_n)||_{L^2(B_r,g)} = ||\nabla\mathcal{E}(u_n - \alpha_n)||_{L^2(B_r,g)} \leqslant ||\mathcal{E}(u_n - \alpha_n)||_{H^1(B_r,g)} \leqslant$$
$$\leqslant ||u_n - \alpha_n||_{H^{1/2}(\partial B_r\setminus\partial M,g)} \leqslant C||u_n - \alpha_n||_{H^1(M\setminus B_r,g)} \leqslant C/n,$$

where in the second and third inequalities we have used in order estimate (7.1) and the trace inequality. We got a contradiction. Hence inequality (2.3) is true.

Note that for any $\rho r < \epsilon$ the first inequality scales as

$$||\mathcal{E}(u)||^2_{L^2(B_{\rho r},g)} \leqslant C||u||^2_{L^2(M\setminus B_{\rho r},g)} + C\rho^2||\nabla u||^2_{L^2(M\setminus B_{\rho r},g)},$$

while the second inequality scales as

$$||\nabla\mathcal{E}(u)||^2_{L^2(B_{\rho r},g)} \leqslant C||\nabla u||^2_{L^2(M\setminus B_{\rho r},g)}.$$

Therefore, $||\mathcal{E}(u)||^2_{H^1(B_{\rho r},g)} \leqslant C||u||^2_{L^2(M\setminus B_{\rho r},g)} + C||\nabla u||^2_{L^2(M\setminus B_{\rho r},g)}$ for ϵ small enough. $\square$

Claim 2. One has

$$\limsup_{\epsilon\to 0} \sigma_k^N(M \setminus B_\epsilon,g) \leqslant \sigma_k(M,g).$$

Proof. We only consider the case of $B_\epsilon \cap \partial M \neq \varnothing$. The case of $B_\epsilon \cap \partial M = \varnothing$ is easier and follows the exactly same arguments. The proof is similar to the proof of [**6**, Theorem 3.5].

Let V_k be a $k-$dimensional subspace of $H^1(M,g)$ and $v \in V_k$ such that

$$\sigma_k(M,g) = \max_{u\in V_k\setminus\{0\}} \frac{\int_M |\nabla u|^2 dv_g}{\int_{\partial M} u^2 ds_g}.$$

Let $u_1,\dots,u_k$ be an orthonormal basis in V_k. We modify the functions $u_i, i = 1,\dots,k$ as

$$u_{i,\epsilon} = u_i - \frac{1}{L(\partial M \setminus \partial B_\epsilon)} \int_{\partial M\setminus\partial B_\epsilon} u_i ds_g.$$

Then $\int_{\partial M\setminus\partial B_\epsilon} u_{i,\epsilon} ds_g = 0$. Consider the space $V_{k,\epsilon} := span(u_{1,\epsilon},\dots,u_{k,\epsilon})$. Since $\dim V_{k,\epsilon} = k$ one has

$$\sigma_k^N(M \setminus B_\epsilon,g) \leqslant \max_{u_\epsilon\in V_{k,\epsilon}\setminus\{0\}} \frac{\int_{M\setminus B_\epsilon} |\nabla u_\epsilon|^2 dv_g}{\int_{\partial M\setminus\partial B_\epsilon} u_\epsilon^2 ds_g}.$$

Moreover, since the dimension of $V_{k,\epsilon}$ is finite then there exists a function $v_\epsilon \in V_{k,\epsilon}$ such that

$$\sigma_k^N(M \setminus B_\epsilon,g) \leqslant \frac{\int_{M\setminus B_\epsilon} |\nabla v_\epsilon|^2 dv_g}{\int_{\partial M\setminus\partial B_\epsilon} v_\epsilon^2 ds_g}. \tag{2.4}$$

Let $v_\epsilon = \sum_{i=1}^k c_i u_{i,\epsilon}$. We build the following function $v = \sum_{i=1}^k c_i u_i \in V_k \subset H^1(M,g)$. Note that $\nabla v_\epsilon = \sum_{i=1}^k c_i \nabla u_{i,\epsilon} = \sum_{i=1}^k c_i \nabla u_i = \nabla v$ on $M \setminus B_\epsilon$. Thus $\int_{M\setminus B_\epsilon} |\nabla v_\epsilon|^2 dv_g = \int_{M\setminus B_\epsilon} |\nabla v|^2 dv_g \to \int_M |\nabla v|^2 dv_g$ as $\epsilon \to 0$. Moreover, it is easy to see that

$$\int_{\partial M\setminus\partial B_\epsilon} v_\epsilon^2 ds_g = \sum_i c_i^2\Big(\int_{\partial M\setminus\partial B_\epsilon} u_i^2 dv_g - \frac{1}{L(\partial M\setminus\partial B_\epsilon,g)}\Big(\int_{\partial M\setminus\partial B_\epsilon} u_i ds_g\Big)^2\Big)+$$
$$+\sum_{i\neq j} 2c_ic_j\Big(\int_{\partial M\setminus\partial B_\epsilon} u_iu_j ds_g - \frac{1}{L(\partial M\setminus\partial B_\epsilon,g)}\int_{\partial M\setminus\partial B_\epsilon} u_i ds_g \int_{\partial M\setminus\partial B_\epsilon} u_j ds_g\Big),$$

which converges to $\int_{\partial M} v^2 ds_g$ as $\epsilon \to 0$. Then (2.4) implies

$$\limsup_{\epsilon\to 0} \sigma_k^N(M\setminus B_\epsilon,g) \leqslant \limsup_{\epsilon\to 0} \frac{\int_{M\setminus B_\epsilon} |\nabla v_\epsilon|^2 dv_g}{\int_{\partial M\setminus\partial B_\epsilon} v_\epsilon^2 ds_g} = \frac{\int_M |\nabla v|^2 dv_g}{\int_{\partial M} v^2 ds_g} \leqslant \sigma_k(M,g).$$

□

Now we are ready to prove the Lemma. The proof is similar to the proof of [**74**, Lemma 3.2]. Let u_ϵ be a normalized σ_k^N-eigenfunction. By Claim 2 u_ϵ are uniformly bounded. If $B_\epsilon \cap \partial M = \varnothing$ then we take the harmonic continuation into B_ϵ. It is known that the operators of harmonic continuation into B_ϵ are uniformly bounded (see [**100**, Example 1]). Otherwise we extend u_ϵ into B_ϵ by $\mathcal{E}(u_\epsilon)$. By Claim 1 these operators are also uniformly bounded. Therefore, we get a uniformly bounded in $H^1(M,g)$ sequence $\{\tilde u_\epsilon\}$. Then there exists $\epsilon_l \to 0$ such that $\tilde u_{\epsilon_l} \rightharpoonup u$ in $H^1(M,g)$. Thus, $\tilde u_{\epsilon_l} \to u$ in $L^2(M,g)$ by the Rellich-Kondrachov embedding theorem. The standard elliptic estimates imply $u_{\epsilon_l} \to u$ in $C^\infty_{loc}(M\setminus\{p\})$. Consider a function $\varphi \in C_c^\infty(M\setminus\{p\})$ such that $supp(\varphi) \subset M\setminus B_R$ for a ball B_R centred at p with R fixed. Extracting a subsequence by Claim 2 one can assume that $\sigma_k^N(M\setminus B_{\epsilon_l},g) \to \sigma$. Then we have

$$\int_M \langle\nabla u, \nabla\varphi\rangle dv_g = \lim_{l\to 0}\int_{M\setminus B_R} \langle\nabla u_{\epsilon_l}, \nabla\varphi\rangle dv_g = \lim_{l\to 0}\sigma_k^N(M\setminus B_{\epsilon_l},g)\int_{M\setminus B_R} u_{\epsilon_l}\varphi dv_g =$$
$$= \sigma\int_M u\varphi dv_g.$$

Hence u is an eigenfunction with eigenvalue σ. Thus all accumulation points of $\{\sigma_k^N(M\setminus B_{\epsilon_l},g)\}$ are in the Steklov spectrum of M. Our aim now is to show that $\sigma = \sigma_k(M,g)$. We will do this by showing that the u is orthogonal in $L^2(\partial M,g)$ to the first $k-1$ Steklov eigenfunctions of (M,g). We use the proof by induction.

Let u_ϵ be a first Steklov-Neumann eigenfunction of $(M\setminus B_\epsilon,g)$. We have already shown that $\tilde u_\epsilon \rightharpoonup u$ in $H^1(M,g)$ then by the trace embedding theorem one has $\tilde u_\epsilon \to u$ in $H^{1/2}(\partial M,g)$ and hence in $L^2(\partial M,g)$. In particular, one has $||u_\epsilon - u||_{L^2(\partial M\setminus\partial B_\epsilon,g)} \to 0$ as $\epsilon\to 0$. Then

$$|\int_{\partial M\setminus\partial B_\epsilon}(u_\epsilon - u)ds_g| \leqslant \int_{\partial M\setminus\partial B_\epsilon}|u_\epsilon - u|ds_g \leqslant L(\partial M\setminus\partial B_\epsilon,g)^{1/2}||u_\epsilon - u||^{1/2}_{L^2(\partial M\setminus\partial B_\epsilon,g)},$$

which converges tp 0 as $\epsilon \to 0$. Since $\int_{\partial M\setminus\partial B_\epsilon} u_\epsilon ds_g = 0$ one then has that $\lim_{\epsilon\to 0}\int_{\partial M\setminus\partial B_\epsilon} u ds_g = \int_{\partial M} u ds_g = 0$. Therefore, u cannot be a constant and since by claim 2 $\limsup_{\epsilon\to 0}\sigma_1^N(M\setminus B_\epsilon,g) = \sigma \leqslant \sigma_1(M,g)$ and σ belongs to the Steklov spectrum of (M,g) we conclude that u is a first Steklov eigenfunction of (M,g) and $\sigma = \sigma_1(M,g)$.

Now suppose that $\limsup_{\epsilon\to 0}\sigma_i^N(M\setminus B_\epsilon,g) = \sigma_i(M,g)$ for any $i<k$. Let u_ϵ be a $k-$th Steklov-Neumann eigenfucntion of $(M\setminus B_\epsilon,g)$. Since $\tilde u_\epsilon \rightharpoonup u$ in $H^1(M,g)$ then the trace embedding theorem implies that $\tilde u_\epsilon \to u$ in $H^{1/2}(\partial M,g)$ in particular $\tilde u_\epsilon \to u$ in $L^2(\partial M,g)$ whence $||u_\epsilon - u||_{L^2(\partial M\setminus\partial B_\epsilon,g)} \to 0$. Let v_ϵ be an $i-$th Steklov-Neumann eigenfunction of $(M\setminus B_\epsilon,g)$ with $i<k$. Then $\int_{\partial M\setminus\partial B_\epsilon} u_\epsilon v_\epsilon ds_g = 0$ moreover we have supposed that v is an $i-$th Steklov eigenfunction of (M,g). One has

$$|\int_{\partial M\setminus\partial B_\epsilon}(u_\epsilon v_\epsilon - uv)ds_g| \leqslant$$

$$\leqslant \int_{\partial M\setminus\partial B_\epsilon}|u_\epsilon v_\epsilon - uv|ds_g = \int_{\partial M\setminus\partial B_\epsilon}|u_\epsilon v_\epsilon - u_\epsilon v + u_\epsilon v - uv|ds_g \leqslant$$

$$\leqslant \int_{\partial M\setminus\partial B_\epsilon}|u_\epsilon(v_\epsilon - v)|ds_g + \int_{\partial M\setminus\partial B_\epsilon}|v(u_\epsilon - u)|ds_g \leqslant$$

$$\leqslant \Big(\int_{\partial M\setminus\partial B_\epsilon} u_\epsilon^2 ds_g\Big)^{1/2}\Big(\int_{\partial M\setminus\partial B_\epsilon}(v_\epsilon - v)^2 ds_g\Big)^{1/2} +$$

$$+\Big(\int_{\partial M\setminus\partial B_\epsilon} v_\epsilon^2 ds_g\Big)^{1/2}\Big(\int_{\partial M\setminus\partial B_\epsilon}(u_\epsilon - u)^2 ds_g\Big)^{1/2} \to 0 \text{ as } \epsilon \to 0.$$

Hence $\int_{\partial M\setminus\partial B_\epsilon} u_\epsilon v_\epsilon ds_g \to \int_{\partial M} uv ds_g$ as $\epsilon\to 0$. But $\int_{\partial M\setminus\partial B_\epsilon} u_\epsilon v_\epsilon ds_g = 0$ for all ϵ. Thus $\int_{\partial M} uv ds_g = 0$. We conclude that u is orthogonal in $L^2(\partial M,g)$ to the first $k-1$ Steklov eigenfunctions. Thus $\sigma = \sigma_k^N(M,g)$.

□

We endow the set of Riemannian metrics on Σ with the $C^\infty-$topology. Then the following "continuity" result holds.

Proposition 2.2. *Let Σ be a surface with boundary and $\Omega\subset\Sigma$ be a Lipschitz domain. Let the sequence of Riemannian metrics g_m on Σ converge in $C^\infty-$topology to the metric g. Then $\sigma_k^*(\Sigma,[g_m]) \to \sigma_k^*(\Sigma,[g])$. Similarly, if $h_m|_{\overline\Omega}$ converge to $g|_{\overline\Omega}$ in C^∞-topology, then $\sigma_k^{N*}(\Omega,\partial^S\Omega,[h_m|_{\overline\Omega}]) \to \sigma_k^{N*}(\Omega,\partial^S\Omega,[g|_{\overline\Omega}])$.*

Proof. We provide a proof for the functional $\sigma_k^*(\Sigma,[g])$. The proof for the functional $\sigma_k^{N*}(\Omega,[g|_{\overline\Omega}])$ follows the exactly same arguments.

Choose any $\varepsilon>0$ and consider m large enough. One has

$$\frac{1}{1+\varepsilon}fg_m(v,v) \leqslant fg(v,v) \leqslant (1+\varepsilon)fg_m(v,v), \quad \forall v\in\Gamma(TM\setminus\{0\}),$$

where f is any positive smooth function on Σ. Then by [**15**, Proposition 32] one has

$$\frac{1}{(1+\varepsilon)^6}\bar{\sigma}_k(\Sigma,fg_m) \leqslant \bar{\sigma}_k(\Sigma,fg) \leqslant (1+\varepsilon)^6\bar{\sigma}_k(\Sigma,fg_m).$$

Taking the supremum over all f yields

$$\frac{1}{(1+\varepsilon)^6}\sigma^*_k(\Sigma,[g_m]) \leqslant \sigma^*_k(\Sigma,[g]) \leqslant (1+\varepsilon)^6\sigma^*_k(\Sigma,[g_m]),$$

which completes the proof since this inequality holds for any $\varepsilon > 0$. $\square$

2.2. Discontinuous metrics

Let Σ be a compact surface with boundary. Consider a set of pairwise disjoint Lipschitz domains $\{\Omega_i\}_{i=1}^s$ in Σ such that $\Sigma = \bigcup_{i=1}^s \overline{\Omega}_i$. Let $C^\infty_+(\Sigma,\{\Omega_i\})$ denote a set of smooth positive functions on $\bigcup_{i=1}^s \overline{\Omega}_i$, i.e. $\rho \in C^\infty_+(\Sigma,\{\Omega_i\})$ means that $\rho|_{\Omega_i} = \rho_i \in C^\infty(\overline{\Omega}_i)$ are positive for every i. Similarly, $C^\infty(\Sigma,\{\Omega_i\})$ denotes a set of continuous functions on $\bigcup_{i=1}^s \overline{\Omega}_i$. We introduce discontinuous metrics on Σ defined as $\rho g \in [g]$, where $\rho \in C^\infty_+(\Sigma,\{\Omega_i\})$ and g is a genuine Riemannian metric. The space $C^k_+(\Sigma,\{\Omega_i\})$ is defined in a similar way. The Steklov spectrum of the metric ρg is defined as the set of critical values of the Rayleigh quotient

$$R_{\rho g}[\varphi] = \frac{\displaystyle\int_\Sigma |\nabla_g\varphi|^2_g dv_g}{\displaystyle\int_{\partial\Sigma} \rho^{\frac{1}{2}}\varphi^2 ds_g}.$$

This is the Rayleigh quotient of the *Steklov problem with density* ρ. The Steklov spectrum with density ρ is well-defined for any non-negative $\rho \in L^\infty(\Sigma,g)$ (see [**63**, Proposition 1.3]). Elliptic regularity implies that the eigenfunctions are at least $1/2-$Hölder continuous on $\partial\Sigma$. Therefore, Steklov eigenvalues of the metric ρg admit the following variational characterization

$$\sigma_k(\Sigma,\rho g) = \inf_{E_{k+1}} \sup_{\varphi\in E_{k+1}} R_{\rho g}[\varphi],$$

where E_{k+1} ranges over all $(k+1)$-dimensional subspaces of $C^0(\Sigma)$.

We introduce the following notation

$$\sigma^*_k(\Sigma,\{\Omega_i\},[g]) = \sup\{\bar{\sigma}_k(\rho g) \mid \rho \in C^\infty_+(\Sigma,\{\Omega_i\})\},$$

where $\bar{\sigma}_k(\rho g)$ is the normalized k-th eigenvalue given by

$$\bar{\sigma}_k(\rho g) = \sigma_k(\rho g)L_{\rho g}(\partial\Sigma).$$

The following lemma particularly asserts that the quantity $\sigma^*_k(\Sigma,\{\Omega_i\},[g])$ is well-defined.

Lemma 2.3. *Let* (Σ,g) *be a Riemannian surface with boundary. Consider a set of pairwise disjoint Lipschitz domains* Ω_i *such that* $\Sigma = \bigcup_{i=1}^s \overline{\Omega}_i$. *Then one has*

$$\sigma_k^*(\Sigma,\{\Omega_i\},[g]) = \sigma_k^*(\Sigma,[g])$$

Proof. The proof follows the same steps as the proof of Lemma 2 in the paper [**34**]. We provide it here.

Since the set of discontinuous metrics is larger than the set of continuous ones, we have $\sigma_k^*(\Sigma,\{\Omega_i\},[g])) \geqslant \sigma_k^*(\Sigma,[g])$. Therefore, we have to prove that

$$\sigma_k^*(\Sigma,\{\Omega_i\},[g])) \leqslant \sigma_k^*(\Sigma,[g]),$$

which is equivalent to

$$\sigma_k(\Sigma,\rho g) \leqslant \sigma_k^*(\Sigma,[g]), \tag{2.5}$$

where $\rho \in C_+^\infty(\Sigma,\{\Omega_i\})$ and $\int_{\partial\Sigma} \rho^{1/2} ds_g = 1$.

Let E_k be the eigenspace corresponding to the k-th Steklov eigenvalue of the metric ρg. We put

$$S = \{u \in H^1(\Sigma,\rho g) \mid u \perp_{L^2(\Sigma,\rho g)} E_0,\ldots,E_{k-1}, \int_{\partial\Sigma} \rho^{1/2} u^2 ds_g = 1\}$$

For any $\varepsilon > 0$ we consider the functional

$$\mathcal{F}_\rho[u] := \int_\Sigma |\nabla_g u|^2 dv_g - (\sigma_k(\Sigma,\rho g) - \varepsilon) \int_{\partial\Sigma} \rho^{1/2} u^2 ds_g.$$

It immediately follows that $\mathcal{F}_\rho[u] \geqslant \varepsilon, \forall u \in S$.

Let $0 < a := \min_{\cup\{\Omega_i\}} \rho$ and $\max_{\cup\{\Omega_i\}} =: b < \infty$. We define a smooth non-decreasing function $\chi(t)$ on $\mathbb{R}_+$ that equals zero if $t < 1/2$ and equals 1 when $t > 1$ and define the following parametrized family of functions:

$$\rho_\delta(x) = \begin{cases} \rho(x) & \text{if } x \notin U \\ \rho(x)\chi\Big(\frac{d^2(x)}{\delta}\Big) + b\Big(1 - \chi\Big(\frac{d^2(x)}{\delta}\Big)\Big) & \text{if } x \in U \end{cases}$$

where d is the distance function from a point $x \in \Sigma$ to $\cup\{\partial\Omega_i \cap \partial\Omega_j\}$, $i \neq j$ and U is a sufficiently small tubular neighborhood of $\cup\{\partial\Omega_i \cap \partial\Omega_j\}$, $i \neq j$ where d^2 is smooth. We have:

- $\Big(\frac{a}{b}\Big)\rho \leqslant \rho_\delta \leqslant \Big(\frac{b}{a}\Big)\rho$;
- $\lim_{\delta\to 0} \int_{\partial\Sigma} \rho_\delta^{1/2} ds_g = 1$;
- $\lim_{\delta\to 0} \int_{\partial\Sigma} |\rho_\delta^{1/2} - \rho^{1/2}|^q ds_g = 0, \forall q < \infty$.

We want to prove that $\mathcal{F}_{\rho_\delta}[u] \geqslant 0, \forall u \in S$.

Consider $T = (\sigma_k(M,\rho g) - \varepsilon)\sqrt{\frac{b}{a}}$ and divide the set S into two parts S_1 and S_2:

$$S_1 := \{u \in S | \int_\Sigma |\nabla_g u|^2 dv_g \geqslant T\},$$
$$S_2 := S \setminus S_1 = \{u \in S | \int_\Sigma |\nabla_g u|^2 dv_g < T\}.$$

If $u \in S_1$ then

$$\mathcal{F}_{\rho_\delta}[u] = \int_\Sigma |\nabla_g u|^2 dv_g - (\sigma_k(\Sigma,\rho g) - \varepsilon)\int_{\partial\Sigma} \rho_\delta^{1/2} u^2 ds_g \geqslant$$
$$\geqslant (\sigma_k(\Sigma,\rho g) - \varepsilon)\Big(\sqrt{\frac{b}{a}} - \int_{\partial\Sigma} \rho_\delta^{1/2} u^2 ds_g\Big) \geqslant (\sigma_k(\Sigma,\rho g) - \varepsilon)\sqrt{\frac{b}{a}}(1 - \int_{\partial\Sigma} \rho^{1/2} u^2 ds_g) = 0.$$

If $u \in S_2$ then

$$\mathcal{F}_{\rho_\delta}[u] = \int_\Sigma |\nabla_g u|^2 dv_g - (\sigma_k(\Sigma,\rho g) - \varepsilon)\int_{\partial\Sigma} \rho_\delta^{1/2} u^2 ds_g =$$
$$= \int_\Sigma |\nabla_g u|^2 dv_g - (\sigma_k(\Sigma,\rho g) - \varepsilon) - (\sigma_k(\Sigma,\rho g) - \varepsilon)\int_{\partial\Sigma} (\rho_\delta^{1/2} - \rho^{1/2}) u^2 ds_g \geqslant$$
$$\geqslant \varepsilon - (\sigma_k(\Sigma,\rho g) - \varepsilon)\Big(\int_{\partial\Sigma} (\rho_\delta^{1/2} - \rho^{1/2})^q ds_g\Big)^{1/q}\Big(\int_{\partial\Sigma} |u|^p ds_g\Big)^{2/p} \geqslant$$
$$\geqslant \varepsilon - (\sigma_k(\Sigma,\rho g) - \varepsilon)\frac{\varepsilon}{\sigma_k(\Sigma,\rho g) - \varepsilon} = 0.$$

In the last inequality we put

$$\Big(\int_{\partial\Sigma} (\rho_\delta^{1/2} - \rho^{1/2})^q ds_g\Big)^{1/q}\Big(\int_{\partial\Sigma} |u|^p ds_g\Big)^{2/p} = \frac{\varepsilon}{\sigma_k(\Sigma,\rho g) - \varepsilon}$$

since $\int_{\partial\Sigma} (\rho_\delta^{1/2} - \rho^{1/2})^q ds_g \to 0$ as $\delta \to 0$ and $\int_{\partial\Sigma} |u|^p ds_g < +\infty$, indeed

$$a^{1/2} ||u||^p_{L^p(\partial\Sigma,g)} \leqslant \int_{\partial\Sigma} |u|^p \rho^{1/2} ds_g = ||u||^p_{L^p(\partial\Sigma,\rho g)} \leqslant$$
$$\leqslant L_{\rho g}(\partial\Sigma)^{\frac{2-p}{2p}} ||u||^2_{L^2(\partial\Sigma,\rho g)} = L_{\rho g}(\partial\Sigma)^{\frac{2-p}{2p}} < +\infty.$$

Hence, $\mathcal{F}_{\rho_\delta}[u] \geqslant 0, \forall u \in S$ which implies $\sigma_k(\Sigma,\rho_\delta g) \geqslant \sigma_k(\Sigma, \rho g) - \varepsilon$. We then have

$$\bar\sigma_k(\Sigma,\rho_\delta g) = \sigma_k(\Sigma,\rho_\delta g) L_{\rho_\delta g}(\partial\Sigma) \geqslant \sigma_k(\Sigma, \rho g) L_{\rho_\delta g}(\partial\Sigma) - \varepsilon L_{\rho_\delta g}(\partial\Sigma).$$

Therefore, $\sigma_k^*(\Sigma, [g]) \geqslant \sigma_k(\Sigma, \rho g) L_{\rho_\delta g}(\partial\Sigma) - \varepsilon L_{\rho_\delta g}(\partial\Sigma)$. Letting $\delta \to 0$ one then gets $\sigma_k^*(\Sigma, [g]) \geqslant \sigma_k(\Sigma, \rho g) - \varepsilon$ that implies (2.5) since ε is arbitrary small. $\square$

Lemma 2.3 implies the following lemma whose proof is postponed to Section 7.2.

Lemma 2.4. *Let (Σ,g) be a Riemannian surface with boundary. Consider a set of pairwise disjoint domains Ω_i such that $\Sigma = \bigcup_{i=1}^s \overline{\Omega}_i$ and $\Omega_i \cap \partial\Sigma = \partial^S\Omega_i$. Let $(\Omega,h) = \sqcup(\overline{\Omega}_i, g|_{\overline{\Omega}_i})$ and $\partial^S\Omega = \sqcup\partial^S\Omega_i$. Then for all $k \geqslant 0$ one has*

$$\sigma_k^*(\Sigma,[g]) \geqslant \sigma_k^{N*}(\Omega,\partial^S\Omega, [h]).$$

2.3. Steklov-Neumann spectrum of a subdomain.

This section is devoted to the following technical lemma

Lemma 2.5. *Let $\rho_\delta \in C^\infty_+(\Sigma,\{\Omega,\Sigma \setminus \Omega\})$ such that $\rho_\delta|_\Omega \equiv 1$ and $\rho_\delta|_{\Sigma\setminus\Omega} \equiv \delta$. Then one has*

$$\liminf_{\delta\to 0} \sigma_k(\rho_\delta g) \geqslant \sigma_k^N(\Omega,\partial^S\Omega, g),$$

where $\sigma_k^{N*}(\Omega,\partial^S\Omega, g)$ is the k-th Steklov Neumann eigenvalue of the domain (Ω,g) and $\partial^S\Omega = \partial\Sigma \cap \Omega \neq \varnothing$.

Proof. The idea of the proof comes from the proof of [**26**, Section 2, Step 2].

Case I. First we consider the case when $\Omega^c \cap \partial\Sigma \neq \varnothing$. Let Ω^c denotes $int(\Sigma \setminus \Omega)$ and $\partial^S\Omega^c = \partial\Omega^c \cap \partial\Sigma$. Since by elliptic regularity eigenfunctions of the Steklov problem with bounded density are in H^1 on the boundary we can restrict ourselves to the space $H^1(\partial\Sigma,g)$. More precisely, let ψ be an eigenfunction with eigenvalue σ then by elliptic regularity:

$$||\psi||^2_{H^1(\partial\Sigma,\rho_\delta g)} \leqslant C(||\sigma\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)} + ||\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}) \leqslant C(\sigma^2+1)||\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}$$

for some positive constant C. This implies

$$\frac{||\nabla\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}}{||\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}} \leqslant C(\sigma^2+1) - 1.$$

More generally we see that if $\varphi \in \operatorname{span}\langle\psi_0,\dots,\psi_k\rangle$, where ψ_i is in the i-th eigenspace of (Σ,g_δ) then there exists a constant $C_k > 0$ such that

$$\frac{||\nabla\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}}{||\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}} \leqslant C_k.$$

Therefore, we set

$$\mathcal{H} := \{\varphi \in H^1(\partial\Sigma,g) \mid \exists\, C_k > 0,\ \frac{||\nabla\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}}{||\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}} \leqslant C_k\},$$

$$\mathcal{H}_1 := \{\varphi \in \mathcal{H} \mid \frac{\partial\hat\varphi}{\partial n} = 0 \text{ on } \partial^S\Omega^c\},$$

where $\hat\varphi$ stands for the harmonic continuation of φ into Σ and

$$\mathcal{H}_2 := \{\varphi \in \mathcal{H} \mid \varphi \in H^1_0(\partial^S\Omega^c,g),\ \varphi_{|\Omega} = 0\}.$$

Claim 1. One has

$$\int_\Sigma \langle\nabla\hat\varphi,\nabla\hat\psi\rangle_{\tilde g}\, dv_{\tilde g} = 0, \forall\varphi\in\mathcal{H}_1, \psi\in\mathcal{H}_2,$$

for any metric $\widetilde g \in [g]$.

Proof.

$$\int_\Sigma \langle\nabla\hat\varphi,\nabla\hat\psi\rangle_{\tilde g}\, dv_{\tilde g} = \int_\Sigma \Delta_{\tilde g}\hat\varphi\hat\psi\, dv_{\tilde g} + \int_{\partial\Sigma}\frac{\partial\hat\varphi}{\partial\tilde n}\psi\, ds_{\tilde g} = \int_{\partial^S\Omega^c}\frac{\partial\hat\varphi}{\partial\tilde n}\psi\, ds_{\tilde g} + \int_{\partial^S\Omega}\frac{\partial\hat\varphi}{\partial\tilde n}\psi\, ds_{\tilde g} = 0.$$

$\square$

Consider the operator $\mathcal{E}$ defined in section 2.1 by

$$\begin{cases} \Delta_g \mathcal{E}(u) = 0 & \text{in } \Sigma, \\ \frac{\partial \mathcal{E}(u)}{\partial n} = 0 & \text{on } \partial^S \Omega^c, \\ \mathcal{E}(u) = u & \text{on } \partial^S \Omega. \end{cases}$$

For a function $\varphi \in H^1(\partial\Sigma, g)$ we fix its decomposition $\varphi_1 + \varphi_2$ with

$$\varphi_1 = \begin{cases} \varphi & \text{on } \partial^S \Omega, \\ \mathcal{E}(\varphi) & \text{on } \partial^S \Omega^c \end{cases}$$

and $\varphi_2 = \varphi_1 - \varphi$. Note that $\hat{\varphi}_1 = \mathcal{E}(\varphi_1)$.

For the sake of simplicity we use the symbols σ_k^δ for $\sigma_k(\rho_\delta g)$, g_δ for $\rho_\delta g$ and R_δ for the Rayleigh quotient

$$R_\delta[\varphi] = \frac{\displaystyle\int_\Sigma |\nabla \hat{\varphi}|^2_{g_\delta} dv_{g_\delta}}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}}.$$

Claim 2. There exists a constant that we also denote by $C_k > 0$ such that $\sigma_k^\delta \leqslant C_k$.

Proof. Theorem 1.1 implies that there exists a constant $C(k) > 0$ such that

$$\sigma_k^*(\Sigma, [g]) \leqslant C(k).$$

By Lemma 2.3 for every δ one has

$$\sigma_k^\delta L_{g_\delta}(\partial\Sigma) \leqslant \sigma_k^*(\Sigma, [g]) \leqslant C(k).$$

Therefore,

$$\sigma_k^\delta \leqslant \frac{C(k)}{L_{g_\delta}(\partial\Sigma)} = \frac{C(k)}{L_g(\partial^S\Omega) + \delta^{1/2} L_g(\partial^S\Omega^c)} \leqslant \frac{C(k)}{L_g(\partial^S\Omega)} = C_k.$$

$\square$

Let W_k be the set of $k+1$-dimensional subspaces of $\mathcal{H}$ satisfying the condition that $R_\delta|_{W_k} \leqslant C_k$. Claim 2 particularly implies that the space spanned by the first $k+1$ eigenfunctions is in W_k, i.e. $W_k \neq \varnothing$.

Claim 3. For every $\varphi \in V \in W_k$ there exists a constant $C > 0$ such that

$$\int_{\partial^S \Omega^c} \varphi_2^2 \, ds_{g_\delta} \leqslant C\sqrt{\delta} \int_{\partial\Sigma} \varphi^2 dv_{g_\delta}.$$

Proof. By Claim 1 one has

$$\int_\Sigma \langle \nabla \hat{\varphi}_1, \nabla \hat{\varphi}_2 \rangle_g dv_g = 0.$$

Further, since $\varphi \in V \in W_k$ we have

$$C_k \geqslant R_\delta[\varphi] = \frac{\displaystyle\int_\Sigma |\nabla \hat{\varphi}|_g^2 dv_g}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} = \frac{\displaystyle\int_\Sigma |\nabla \hat{\varphi}_1|^2 dv_g + \int_\Sigma |\nabla \hat{\varphi}_2|_g^2 dv_g}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \geqslant$$

$$\geqslant \frac{\displaystyle\int_{\Omega^c} |\nabla \hat{\varphi}_2|_g^2 dv_g}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} = \frac{1}{\delta^{1/2}} \frac{\displaystyle\int_{\Omega^c} |\nabla \hat{\varphi}_2|_g^2 dv_g}{\displaystyle\int_{\partial^S\Omega^c} \varphi_2^2 ds_g} \frac{||\varphi_2||^2_{L^2(\partial^S\Omega^c, g_\delta)}}{||\varphi||^2_{L^2(\partial\Sigma, g_\delta)}} \geqslant \frac{\sigma_1^D(\Omega^c, \partial^S\Omega^c, g)}{\sqrt{\delta}} \frac{||\varphi_2||^2_{L^2(\partial^S\Omega^c, g_\delta)}}{||\varphi||^2_{L^2(\partial\Sigma, g_\delta)}},$$

where $\sigma_1^D(\Omega^c, \partial^S\Omega^c, g)$ is the first non-zero Steklov-Dirichlet eigenvalue of (Ω^c, g) (see [**4**]). □

Claim 4. For every $\varphi \in V \in W_k$ and for every sufficiently small δ there exists a constant $C > 0$ such that

$$\int_{\partial\Sigma} \varphi^2 \; ds_{g_\delta} \leqslant (1 + C\delta^{1/4}) \int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta}.$$

Proof. One has

$$||\varphi||^2_{L^2(\partial\Sigma, g_\delta)} = \int_{\partial^S\Omega^c} (\varphi_1 + \varphi_2)^2 dv_{s_\delta} + \int_{\partial^S\Omega} \varphi_1^2 ds_{g_\delta} \leqslant$$
$$\leqslant \left(1 + \frac{1}{\varepsilon}\right) \int_{\partial\Sigma} \varphi_2^2 ds_{g_\delta} + (1+\varepsilon) \int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta},$$

for every $\varepsilon > 0$. Applying Claim 3 we obtain

$$||\varphi||^2_{L^2(\partial\Sigma, g_\delta)} \leqslant C\sqrt{\delta}\left(1 + \frac{1}{\varepsilon}\right) \int_{\partial\Sigma} \varphi^2 ds_{g_\delta} + (1+\varepsilon) \int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta},$$

and hence,

$$\left(1 - C\sqrt{\delta}\left(1 + \frac{1}{\varepsilon}\right)\right) ||\varphi||^2_{L^2(\partial\Sigma, g_\delta)} \leqslant (1+\varepsilon) ||\varphi_1||^2_{L^2(\partial\Sigma, g_\delta)}.$$

Choosing $\varepsilon = \delta^{1/4}$ completes the proof. □

Claim 5. For every $\varphi \in V \in W_k$ and for every sufficiently small δ there exists a constant $C > 0$ such that

$$\int_{\partial^S\Omega^c} \varphi_1^2 \; ds_g \leqslant C \int_{\partial^S\Omega} \varphi_1^2 ds_g.$$

Proof.

$$C_k \geqslant \frac{\displaystyle\int_{\partial\Sigma} |\nabla \varphi|_{g_\delta}^2 dv_{g_\delta}}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \geqslant \frac{\displaystyle\int_{\partial^S\Omega} |\nabla \varphi|_g^2 ds_g}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} = \frac{\displaystyle\int_{\partial^S\Omega} |\nabla \varphi_1|_g^2 ds_g}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}},$$

since $\varphi = \varphi_1$ on $\partial^S\Omega$. Then by Claim 4 one has

$$C_k \geqslant \frac{\displaystyle\int_{\partial^S\Omega} |\nabla \varphi_1|_g^2 ds_g}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \geqslant \frac{1}{1 + C\delta^{1/4}} \frac{\displaystyle\int_{\partial^S\Omega} |\nabla \varphi_1|_g^2 ds_g}{\displaystyle\int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta}},$$

which implies

$$\begin{aligned}\int_{\partial^S\Omega}|\nabla\varphi_1|_g^2 ds_g \leqslant C_k(1+C\delta^{1/4})\int_{\partial\Sigma}\varphi_1^2 ds_{g_\delta} = \\ = C_k(1+C\delta^{1/4})\Big(\int_{\partial^S\Omega}\varphi_1^2 ds_g + \delta^{1/2}\int_{\partial^S\Omega^c}\varphi_1^2 ds_g\Big).\end{aligned} \tag{2.6}$$

For the rest of the proof C stands for any positive constant depending possibly on Σ and g but not on δ or φ.

Applying in order the trace theorem, estimate (7.1), the Sobolev embedding and inequality (2.6) yield

$$\begin{aligned}||\varphi_1||^2_{L^2(\partial^S\Omega^c,g)} \leqslant C||\hat\varphi_1||^2_{H^1(\Sigma,g)} \leqslant C||\varphi_1||^2_{H^{1/2}(\partial^S\Omega,g)} \leqslant \\ \leqslant C||\varphi_1||^2_{H^1(\partial^S\Omega,g)} = C(||\varphi_1||^2_{L^2(\partial^S\Omega,g)} + ||\nabla\varphi_1||^2_{L^2(\partial^S\Omega,g)}) \leqslant \\ \leqslant C(1+C\delta^{1/4})\Big(||\varphi_1||^2_{L^2(\partial^S\Omega,g)} + \delta^{1/2}||\varphi_1||^2_{L^2(\partial^S\Omega^c,g)}\Big),\end{aligned}$$

which implies the required inequality for δ small enough. □

Further by the fact that $\int_\Sigma\langle\nabla\hat\varphi_1,\nabla\hat\varphi_2\rangle_g dv_g = 0$ and by claim 4 for every $\varphi \in V \in W_k$ and one has

$$\begin{aligned}R_\delta[\varphi] = \frac{\int_\Sigma|\nabla\hat\varphi|_g^2 dv_g}{\int_{\partial\Sigma}\varphi^2 ds_{g_\delta}} = \frac{\int_\Sigma|\nabla\hat\varphi_1|_g^2 dv_g + \int_\Sigma|\nabla\hat\varphi_2|_g^2 dv_g}{\int_{\partial\Sigma}\varphi^2 ds_{g_\delta}} \geqslant \\ \geqslant \frac{1}{1+C\delta^{1/4}}\frac{\int_\Sigma|\nabla\hat\varphi_1|_g^2 dv_g + \int_\Sigma|\nabla\hat\varphi_2|_g^2 dv_g}{\int_{\partial\Sigma}\varphi_1^2 ds_{g_\delta}} \geqslant \\ \geqslant \frac{1}{1+C\delta^{1/4}}\frac{\int_\Sigma|\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial\Sigma}\varphi_1^2 ds_{g_\delta}} = \frac{1}{1+C\delta^{1/4}}\frac{\int_\Sigma|\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial^S\Omega}\varphi_1^2 dv_g + \delta^{1/2}\int_{\partial^S\Omega^c}\varphi_1^2 dv_g}\end{aligned}$$

and by claim 5 we get

$$\begin{aligned}R_\delta[\varphi] \geqslant \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)}\frac{\int_\Sigma|\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial^S\Omega}\varphi_1^2 ds_g} \geqslant \\ \geqslant \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)}\frac{\int_\Omega|\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial^S\Omega}\varphi_1^2 ds_g} \geqslant \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)}R^N_{(\Omega,\partial^S\Omega,g)}[\varphi_{|\Omega}].\end{aligned}$$

where $R^N_{(\Omega,\partial^S\Omega,g)}$ denotes the Rayleigh quotient for the Steklov-Neumann problem in the domain (Ω,g).

Let $V = \operatorname{span}\langle \psi_0, \dots, \psi_k \rangle$, where ψ_i is in the i-th eigenspace of (Σ,g_δ). Then

$$\begin{aligned}\sigma_k^\delta = \max_{\varphi \in V} R_\delta[\varphi] \geqslant \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)} \max_{\varphi \in V} R^N_{(\Omega,\partial^S\Omega,g)}[\varphi|_\Omega] \geqslant \\ \geqslant \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)} \sigma_k^N(\Omega,\partial^S\Omega, g),\end{aligned} \tag{2.7}$$

since the restriction to Ω of the functions ψ_i form the space of the same dimension by unique continuation. Finally, passing to the $\liminf$ as $\delta \to 0$ in (2.7) yields the lemma.

Case II. The case when $\Omega^c \cap \partial\Sigma = \varnothing$ is trivial. Indeed, in this case we have $\partial^S\Omega = \partial\Sigma$. Then for any function φ one has

$$R_\delta[\varphi] = \frac{\displaystyle\int_\Sigma |\nabla \hat\varphi|_g^2 dv_g}{\displaystyle\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \geqslant \frac{\displaystyle\int_\Omega |\nabla \hat\varphi|_g^2 dv_g}{\displaystyle\int_{\partial^S\Omega} \varphi^2 ds_g} = R^N_{(\Omega,\partial^S\Omega,g)}[\varphi|_\Omega].$$

Therefore, considering $V = \operatorname{span}\langle \psi_0, \dots, \psi_k \rangle$, where ψ_i is in the i-th eigenspace of (Σ,g_δ) yields

$$\sigma_k^\delta = \max_{\varphi \in V} R_\delta[\varphi] \geqslant \max_{\varphi \in V} R^N_{(\Omega,\partial^S\Omega,g)}[\varphi|_\Omega] \geqslant \sigma_k^N(\Omega,\partial^S\Omega, g).$$

Taking $\liminf$ as $\delta \to 0$ completes the proof. $\square$

Lemma 2.5 is the key ingredient in the proof of the following proposition. We postpone the proof to Section 7.2.

Proposition 2.6. *Let (Σ,g) be a Riemannian surface with boundary, $\Omega \subset \Sigma$ a Lipschitz domain and $\partial^S\Omega = \partial\Sigma \cap \Omega \neq \varnothing$. Then for all k one has*

$$\sigma_k^*(\Sigma,[g]) \geqslant \sigma_k^{N*}(\Omega,\partial^S\Omega, [g|_{\overline{\Omega}}]).$$

Similarly, let (Σ,g) be a Riemannian surface whose boundary. Let $\partial^S\Sigma$ denote all boundary components with the Steklov boundary condition and $\Omega \subset \Sigma$ be a Lipschitz domain such that $\partial^S\Omega \subset \partial^S\Sigma$. Then for all k one has

$$\sigma_k^{N*}(\Sigma,\partial^S\Sigma, [g]) \geqslant \sigma_k^{N*}(\Omega,\partial^S\Omega, [g|_{\overline{\Omega}}]).$$

As a corollary of Proposition 2.6 we get

Corollary 2.1. *Let (M,g) be a compact Riemannian surface with boundary. Consider a sequence $\{K_n\}$ of smooth domains $K_n \subset M$ such that*

- *$K_r \subset K_s$ $\forall r > s$;*
- *$\cap_n K_n = \{p_1, \dots ,p_l\}$ for some points $p_1, \dots ,p_l \in M$.*

Then one has

$$\lim_{n\to\infty} \sigma_k^{N*}(M \setminus K_n, \partial M \setminus \partial K_n, [g]) = \sigma_k^*(M,[g]).$$

The proof is postponed to Section 7.2.

2.4. Disconnected surfaces.

The proofs of two lemmas below follow the exactly same arguments as the proofs of Lemma 4.9 and Lemma 4.10 in [**55**]. Their proofs are postponed to Section 7.2.

Lemma 2.7. *Let* $(\Omega,g) = \sqcup_{i=1}^{s}(\Omega_i,g_i)$ *be a disjoint union of Riemannian surfaces with Lipschitz boundary. Set* $\partial^S\Omega = \sqcup_{i=1}^{s}\partial^S\Omega_i$. *Then for all* $k > 0$ *one has*

$$\sigma_k^{N*}(\Omega,\partial^S\Omega,[g]) = \max_{\sum\limits_{i=1}^{s} k_i=k,\ k_i>0} \sum_{i=1}^{s}\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i]).$$

Lemma 2.8. *Let* (Σ,g) *be a Riemannian surface with boundary. Consider a set of pairwise disjoint Lipschitz domains* $\{\Omega_i\}_{i=1}^{s}$ *in* Σ *such that* $\Sigma = \bigcup_{i=1}^{s}\overline{\Omega}_i$ *and* $\Omega_i \cap \partial\Sigma = \partial^S\Omega_i \neq \varnothing$ *for* $1 \leqslant i \leqslant s'$. *Then one has*

$$\sigma_k^*(\Sigma,[g]) \geqslant \max_{\sum_{i=1}^{s'} k_i=k,\ k_i\geqslant 0} \sum_{i=1}^{s'}\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g]).$$

3. Proof of Theorem 1.2.

The proof is inspired by the methods of the papers [**109, 39, 51**]. Let Σ be a non-orientable compact surface of genus γ and l boundary components. We pass to its orientable cover $\pi\colon \widetilde{\Sigma} \to \Sigma$. Note that Σ is of genus γ and has $2l$ boundary components. Let τ denote the involution exchanging the sheets of π. If h is a metric on Σ then $g := \pi^*h$ is a metric on $\widetilde{\Sigma}$ invariant with respect to τ, i.e. τ is an isometry of g. Let $\mathcal{D}_{\widetilde{\Sigma}}$ be the Dirichlet-to-Neumann map acting on functions on $\widetilde{\Sigma}$. Then $\tau \circ \mathcal{D}_{\widetilde{\Sigma}} = \mathcal{D}_{\widetilde{\Sigma}} \circ \tau$ and hence Steklov eigenfunctions are divided into $\tau-$odd and $\tau-$even ones. The corresponding Steklov eigenvalues are also divided into odd and even ones. Let $\sigma_k^{\tau}(\widetilde{\Sigma},g)$ the $k-$th $\tau-$even Steklov eigenvalue. Then $\sigma_k^{\tau}(\widetilde{\Sigma},g) = \sigma_k(\Sigma,h)$.

By a well-known theorem of Ahlfors [**2**] there exists a proper conformal branched cover $\psi\colon (\widetilde{\Sigma},g) \to (\mathbb{D}^2,g_{can})$. The word "proper" means $\psi(\partial\widetilde{\Sigma}) = \mathbb{S}^1$. Let d be its degree. Define the following pushed-forward metric g^* on $\mathbb{D}^2$: consider a neighbourhood U of a non-branching point $p \in \mathbb{D}^2$. Its pre-image is a collection of d neighbourhoods $U_i, i = 1,\cdots,d$ on $\widetilde{\Sigma}$. Moreover, $\psi_i := \psi_{|U_i}\colon U_i \to U$ is a diffeomorphism. Then the metric g^* is defined on U as $\sum(\psi_i^{-1})^*g$. The metric g^* is a metric on $\mathbb{D}^2$ with isolated conical singularities at branching points of ψ. The following lemma is trivial

Lemma 3.1. *For any function* $u \in C^\infty(\mathbb{D}^2)$ *one has*

$$\int_{\mathbb{S}^1} u dv_{g^*} = \int_{\partial\widetilde{\Sigma}}(\psi^*u)dv_g$$

and

$$d\int_{\mathbb{D}^2} |\nabla_{g^*}u|^2 dv_{g^*} = \int_{\widetilde{\Sigma}} |\nabla_g(\psi^*u)|^2 dv_g.$$

Further, suppose that there exists an involution ι of $\mathbb{D}^2$ such that

$$\psi \circ \tau = \iota \circ \psi. \tag{3.1}$$

Lemma 3.2. *The involution ι is an isometry of $(\mathbb{D}^2,g^*)$.*

Proof. Indeed, let the neighbourhood $U \subset \mathbb{D}^2$ be small enough and do not contain branching points. Then $\psi^{-1}(U) = \sqcup_{i=1}^d U_i$ and applying τ one gets: $\tau(\psi^{-1}(U)) = \sqcup_{i=1}^d \tau(U_i)$. Note that condition (3.1) implies $\tau(\psi^{-1}(U)) = \psi^{-1}(\iota(U))$. Whence $\psi^{-1}(\iota(U)) = \sqcup_{i=1}^d \tau(U_i)$. Let $\widetilde{\psi_i} := \psi_{\tau(U_i)}$. Then on U one has

$$g^* = \sum_{i=1}^d (\widetilde{\psi_i}^{-1})^* g = \sum_{i=1}^d (\widetilde{\psi_i}^{-1})^* \tau^* g = \sum_{i=1}^d (\widetilde{\psi_i}^{-1} \circ \tau)^* g = \sum_{i=1}^d (\iota \circ \widetilde{\psi_i}^{-1})^* g =$$
$$= \sum_{i=1}^d \iota^* (\widetilde{\psi_i}^{-1})^* g = \iota^* g^*.$$

□

Consider a $j-$th $\iota-$even eigenfunction u_j on $(\mathbb{D}^2,g^*)$ with corresponding eigenvalue $\sigma_j^\iota(\mathbb{D}^2,g^*)$. Then the function $\psi^* u_j$ on $\widetilde{\Sigma}$ is $\tau-$even and hence it projects to a well-defined function v_j on Σ. We can construct the following function $v = \sum_{j=0}^{k-1} c_j v_j$. Note that $\pi^* v = \sum_{j=0}^{k-1} c_j \psi^* u_j = \psi^* u$, where $u := \sum_{j=0}^{k-1} c_j u_j$. Further, let w_i denote an $i-$th eigenfunction on Σ with eigenvalue $\sigma_i(\Sigma,h)$. It is easy to see that one can always find some coefficients $c_0, \cdots ,c_{k-1}$ such that $\int_{\partial\Sigma} v w_i dv_h = 0, i = 0, \cdots , k-1$. Then we can use v as a test function for $\sigma_k(\Sigma,h)$:

$$\sigma_k(\Sigma,h) \leqslant \frac{\int_\Sigma |\nabla_h v|^2 dv_h}{\int_{\partial\Sigma} v^2 dv_h} = \frac{\int_{\widetilde{\Sigma}} |\nabla_g \psi^* u|^2 dv_g}{\int_{\partial\widetilde{\Sigma}} (\psi^* u)^2 dv_g} = d\frac{\int_{\mathbb{D}^2} |\nabla_{g^*} u|^2 dv_{g^*}}{\int_{\mathbb{S}^1} u^2 dv_{g^*}} = d\sigma_k^\iota(\mathbb{D}^2,g^*),$$

where we used Lemma 3.1. Moreover, the second identity in Lemma 3.1 implies $L_{g^*}(\mathbb{S}^1) = L_g(\partial\widetilde{\Sigma}) = 2L_h(\partial\Sigma)$. Whence

$$\overline{\sigma}_k(\Sigma,h) \leqslant \frac{d}{2}\sigma_k^\iota(\mathbb{D}^2,g^*)L_{g^*}(\mathbb{S}^1). \tag{3.2}$$

Consider a conformal map ψ between surfaces with involution ψ: $(\widetilde{\Sigma}, \tau) \to (\mathbb{D}^2, \iota)$ of minimal degree d. The map ψ is conformal, moreover every involution exchanging the orientation on $\mathbb{D}^2$ is conjugate to the involution $\iota_0(z) := \bar{z}$, where we identify $\mathbb{D}^2$ with the unit disc on the complex plane. Therefore, without loss of generality we can assume that $\iota = \iota_0$. The fixed point set of ι_0 is the diameter $\{z \in \mathbb{D}^2 \mid Re(z) = 0\}$. Let $H\mathbb{D}^2$ denote a half-disc for example the right one and $\partial^S H\mathbb{D}^2$ is the right half-circle. Thus, $\sigma_k^{\iota_0}(\mathbb{D}^2,g^*) = \sigma_k^N(H\mathbb{D}^2,\partial^S H\mathbb{D}^2, g^*)$ and inequality (3.2) implies:

$$\begin{aligned}\overline{\sigma}_k(\Sigma,h) &\leqslant \frac{d}{2}\sigma_k^\iota(\mathbb{D}^2,g^*)L_{g^*}(\mathbb{S}^1) = d\overline{\sigma}_k^N(H\mathbb{D}^2,\partial^S H\mathbb{D}^2, g^*) \leqslant \\ &\leqslant d\sigma_k^{N*}(H\mathbb{D}^2,\partial^S H\mathbb{D}^2, [g^*]) \leqslant d\sigma_k^*(\mathbb{D}^2, [g_{can}]) = 2\pi k d,\end{aligned} \tag{3.3}$$

where in the last inequality we used Lemma 2.6 and the fact that there exists a unique up to an isometry conformal class $[g_{can}]$ on $\mathbb{D}^2$. We want to estimate d in formula (3.3). It is known that there exists a proper conformal branched cover $f\colon(\widetilde{\Sigma},g)\to(\mathbb{D}^2,g_{can})$ of degree $d'\leqslant\gamma+2l$ (see [**35**]). One can construct the following function $F(x):=\frac{f(x)+\bar f(\tau(x))}{2}$. Note that $\bar F(x)=F(\tau(x))=\iota(F(x))$ and hence $\iota=\iota_0$. Moreover the degree of F is not greater than $2d'=2(\gamma+2l)$. I.e. there exists a map between $(\widetilde{\Sigma},\tau)$ and $(\mathbb{D}^2,\iota_0)$ of degree not exceeding $2d'=2(\gamma+2l)$ satisfying (3.1). Inequality (3.3) then implies

$$\overline{\sigma}_k(\Sigma,h)\leqslant 4\pi k(\gamma+2l).$$

4. Geometric background

The aim of this section is the proof of Theorem 1.7. For this purpose we provide a necessary background concerning the geometry of moduli space of conformal classes on a surface with boundary. We start with closed orientable surfaces.

4.1. Closed orientable surfaces

Let us recall the *Uniformization theorem.*

Theorem 4.1. *Let Σ be a closed surface and g be a Riemannian metric on it. Then in the conformal class $[g]$ there exists a unique (up to an isometry) metric h of constant Gauss curvature and fixed area. The area assumption is unnecessary in the case of the torus for which we fix the volume of h to be equal to* 1

Remark 4.2. *It follows from the Gauss-Bonnet theorem that the metric h in the Uniformization theorem is of Gauss curvature* 1 *in the case of the sphere,* 0 *in the case of the torus and* -1 *in the rest cases.*

Recall that a Riemannian metric h of constant Gaussian curvature -1 is called *hyperbolic* and a Riemannian surface (Σ,h) endowed with a hyperbolic metric h is called *a hyperbolic surface.* Note also that a hyperbolic surface is necessarily of negative Euler characteristic. We also say that the torus endowed with a metric of curvature $h=0$ is a flat torus and the sphere endowed with the metric $h=1$ is the standard (round) sphere.

4.2. Hyperbolic surfaces

We recall that a *pair of pants* is a compact surface of genus 0 with 3 boundary components. The following theorem plays an underlying role in the theory of hyperbolic surfaces.

Theorem 4.3 (Collar theorem (see e.g. [**8**]))**.** *Let (Σ,h) be an orientable compact hyperbolic surface of genus $\gamma\geqslant 2$ and let $c_1,c_2,\dots,c_m$ be pairwise disjoint simple closed geodesics on (Σ,h). Then the following holds*

- $m\leqslant 3\gamma-3$.

- *There exist simple closed geodesics $c_{m+1},\dots,c_{3\gamma-3}$ which, together with $c_1,\dots,c_m$, decompose Σ into pairs of pants.*
- *The collars*
$$\mathcal{C}(c_i)=\{p\in\Sigma \mid dist(p,c_i)\leqslant w(c_i)\}$$
of widths
$$w(c_i)=\frac{\pi}{l(c_i)}\left(\pi-2\arctan\left(\sinh\frac{l(c_i)}{2}\right)\right)$$
are pairwise disjoint for $i=1,\dots,3\gamma-3$.
- *Each $\mathcal{C}(c_i)$ is isometric to the cylinder $\{(t,\theta)| -w(c_i)<t<w(c_i),\theta\in\mathbb{R}/2\pi\mathbb{Z}\}$ with the Riemannian metric*
$$\left(\frac{l(c_i)}{2\pi\cos\left(\frac{l(c_i)}{2\pi}t\right)}\right)^2\left(dt^2+d\theta^2\right).$$

The decomposition of (Σ,h) into pair of pants which we denote by $\mathcal{P}$ is called *the pants decomposition*. We also say that the geodesics $c_1,\dots,c_{3\gamma-3}$ form $\mathcal{P}$.

4.3. Convergence of hyperbolic metrics

We endow the set of hyperbolic metrics on a given surface Σ with $C^\infty-$topology. In this section we describe the convergence on this topological set which is called *the moduli space of conformal classes* on Σ. Essentially, two cases can happen: the injectivity radii of a sequence of hyperbolic metrics do not go to 0 or they do. The first case is described by *Mumford's compactness theorem* and the second one is treated by *the Deligne-Mumford compactification.*

Proposition 4.4 (Mumford's compactness theorem (see e.g. [**45**]))**.** *Let $\{h_n\}$ be a sequence of hyperbolic metrics on a surface Σ of genus $\geqslant 2$. Assume that the injectivity radii $inj(\Sigma,h_n)$ satisfy $\limsup_{n\to\infty} inj(\Sigma,h_n)>0$. Then there exists a subsequence $\{h_{n_k}\}$, sequence $\{\Phi_k\}$ of smooth automorphisms of Σ and a hyperbolic metric h_∞ on Σ such that the sequence of hyperbolic metrics $\{\Phi_k^* h_{n_k}\}$ converges in C^∞-topology to h_∞.*

If $\lim_{n\to\infty} inj(\Sigma,h_n)=0$ then we say that the sequence $\{h_n\}$ *degenerates.* The thick-thin decomposition implies that if the sequence $\{h_n\}$ degenerates then for each n there exists a collection $\{c_1^n,\dots,c_s^n\}$ of disjoint simple closed geodesics in (Σ,h_n) whose lengths tend to 0 and the length of any geodesic in the complement $\Sigma_n=\Sigma\backslash(c_1^n\cup\ldots\cup c_s^n)$ is bounded from below by a constant independent of n. We call the geodesics $\{c_1^n,\dots,c_s^n\}$ "pinching" or "collapsing". The surface (Σ_n,h_n) is possibly a disconnected hyperbolic surface with geodesic boundary. Moreover, up to a choice of a subsequence all components of Σ_n have the same topological type. Let $\widehat{\Sigma_\infty}$ denote the surface having the same connected components as Σ_n, but with boundary component replaced by marked points. Note that each sequence $\{c_i^n\}$ corresponds to a pair of marked points $\{p_i,q_i\}$ on $\widehat{\Sigma_\infty}$, $i = 1,\dots,s$. Then the punctured surface $\widehat{\Sigma_\infty}\backslash\{p_1,q_1,\dots,p_s,q_s\}$ that we denote by Σ_∞ admits the unique hyperbolic metric h_∞ with cusps at punctures. Now we are ready to formulate one of the underlying results in the theory of *moduli spaces of Riemann surfaces.*

Proposition 4.5 (Deligne-Mumford compactification (see e.g. [**45**])). *Let (Σ,h_n) be a sequence of hyperbolic surfaces such that $inj(\Sigma,h_n) \to 0$. Then up to a choice of subsequence, there exists a sequence of diffeomorphisms $\Psi_n : \Sigma_\infty \to \Sigma_n$ such that the sequence $\{\Psi_n^* h_n\}$ of hyperbolic metrics converges in C^∞_{loc}-topology to the complete hyperbolic metric h_∞ on Σ_∞. Furthermore, there exists a metric of locally constant curvature $\widehat{h_\infty}$ on $\widehat{\Sigma_\infty}$ such that its restriction to Σ_∞ is conformal to h_∞.*

We call $(\widehat{\Sigma_\infty},\widehat{h_\infty})$ a *limiting space* of the sequence (Σ,h_n). We also say that the limit of conformal classes $[h_n]$ is the conformal class $[\widehat{h_\infty}]$ on $\widehat{\Sigma_\infty}$.

Remark 4.6. *We emphazise that $\widehat{h_\infty}$ has* locally *constant curvature, since $\widehat{\Sigma_\infty}$ is possibly disconnected and different connected components could have different signs of Euler characteristic.*

4.4. Orientable surfaces with boundary of negative Euler characteristic

Our exposition of this topic essentially follows the book [**50**].

Let Σ be an orientable surface of genus γ with l boundary components. Consider its *Schottky double* Σ^d defined in following way. We identify Σ with another copy Σ' of Σ with opposite orientation along the common boundary. We get a closed oriented surface of genus $2\gamma + l - 1$. For example the Schottky double of the disk is the sphere and the Schottky double of the cylinder is the torus. In the rest cases we always get a hyperbolic surface as the Schottky double. We endow the surface Σ with a metric g. The next theorem plays a role of the Uniformization theorem for surfaces with boundary.

Proposition 4.7 ([**90**]). *In the conformal class $[g]$ of a metric g on the surface Σ there exists a unique (up to an isometry) metric of constant Gauss curvature and geodesic boundary. More precisely, this metric is of curvature 1 in the case of $\mathbb{D}^2$, of the curvature 0 in the case of the cylinder and of curvature -1 in the rest cases.*

Denote the metric of constant Gauss curvature and geodesic boundary from Theorem 4.7 by h. Consider a Riemannian surface with boundary (Σ,h). Its Schottky double admits the metric h^d defined as $h^d_{|_\Sigma} = h$ and $h^d_{|_{\Sigma'}} = h$. It is a metric of constant curvature and the involution $\iota : \Sigma^d \to \Sigma^d$ that interchanges Σ and Σ' becomes an isometry with $\partial\Sigma$ as the fixed set. Moreover, $(\Sigma,h_n) = (\Sigma^d,h^d_n)/\iota$.

Theorem 4.7 also says that the set of conformal classes on the surface Σ with boundary is in one-to-one correspondence with the set of metrics of constant Gauss curvature and geodesic boundary which is in the one-to-one correspondence with the set of "symmetric" metrics (metrics that go to themselves under the involution ι) of constant curvature on the Schottky double. We endow the set of metrics of constant Gauss curvature and geodesic boundary with $C^\infty-$topology. Consider a sequence of conformal classes $\{c_n\}$ on Σ. It determines uniquely a sequence of "symmetric" metrics of constant curvature $\{h_n^d\}$ on Σ^d. For this sequence we have the same dichotomy as we have seen in the previous sections. Precisely, either $inj(\Sigma^d,h_n^d) \nrightarrow 0$ or $inj(\Sigma^d,h_n^d) \to 0$. In the first case we get a genuine Riemannian metric on Σ^d which is obviously "symmetric" and of constant curvature while in the second case one can find a set of simple closed geodesics $\{c_1^n, \ldots, c_s^n)$ where $s \leqslant 6\gamma+3l-6$ whose lengths $l_{h_n^d}(c_i^n) \to 0$. For the geodesics c_i^n there exist two possibilities: either $\iota(c_i^n) = c_i^n$ or $\iota(c_i^n) = c_j^n$ with $j \neq i$. The first possibility implies that the geodesic c_i^n crosses $\partial\Sigma$ which corresponds to two situations as well: either c_i^n has exactly two points of intersection with $\partial\Sigma$ or it belongs to $\partial\Sigma$, i.e. it is one of the boundary components. The second possibility implies that c_i^n does not crosse $\partial\Sigma$. Taking quotient by ι we then get three types of pinching geodesics on (Σ,h_n) with $inj(\Sigma,h_n) \to 0$: pinching boundary components, pinching simple geodesics which have exactly two points of intersection with the boundary and pinching simple closed geodesics which do not cross the boundary.

4.5. Non-orientable surface with boundary of negative Euler characteristic

Let Σ be a compact non-orientable surface with l boundary components. Note that the Uniformization Theorem 4.7 also holds for non-orientable surfaces. Pick a metric h of constant Gauss curvature and geodesic boundary. We pass to the orientable cover that we denote by $\tilde{\Sigma}$. The surface $\tilde{\Sigma}$ is a compact orientable surface with $2l$ boundary components. The pull-back of the metric h that we denote by $\tilde{h}$ is a metric of constant Gauss curvature and with geodesic boundary. Moreover, this metric is invariant under the involution changing the orientation on $\tilde{\Sigma}$. Consider a sequence $\{h_n\}$ on Σ of metrics of constant Gauss curvature and geodesic boundary such that $inj(\Sigma,h_n) \to 0$ as $n \to \infty$. This sequence corresponds to the sequence $\{\tilde{h}_n\}$ on $\tilde{\Sigma}$ such that $inj(\tilde{\Sigma},\tilde{h}_n) \to 0$ as $n \to \infty$. As we discussed in the previous section for the sequence $\{\tilde{h}_n\}$ one can find pinching geodesics of the following three types: pinching boundary components, pinching simple geodesics which have exactly two points of intersection with the boundary and pinching simple closed geodesics. Consider the geodesics of the third type. For every such geodesic there are two possible cases: either this geodesic maps to itself under the involution changing the orientation or it mappes to another simple closed geodesic which does not cross the boundary. Then taking the quotient by the involution changing the orientation we get two types of simple closed geodesics on Σ which do not crosse the boundary: *one-sided geodesics* which are the images of the geodesics described in the first case and *two-sided geodesics* which are the images of the geodesics described in the second case. The collars of one-sided geodesics are nothing but Möbius bands while the collars of two-sided geodesics are cylinders. Therefore, if $inj(\Sigma,h_n) \to 0$ as $n \to \infty$ then one can find pinching geodesics of the following types: pinching boundary components, pinching simple geodesics which have exactly two points of intersection with the boundary, one-sided pinching simple closed geodesics not crossing the boundary and two-sided pinching simple closed geodesics not crossing the boundary.

4.6. Surfaces with boundary of non-negative Euler characteristic

Here we consider the cases of the disc, the cylinder $\mathcal{C}$ and the Möbius band $\mathbb{MB}$.

It is known that the disc has a unique conformal class (up to an isometry). We denote this conformal class as $[g_{can}]$ or c_{can}, where g_{can} is the flat metric on the disc $\mathbb{D}^2$ with unit boundary length.

Accordingly to Theorem 4.7 in a conformal class on $\mathcal{C}$ there exists a flat metric with geodesic boundary, i.e. a metric on the right circular cylinder. This metric is unique if we fix the length of the boundary. The right circular cylinder is uniquely determined by its height. Therefore, conformal classes on $\mathcal{C}$ are in one-to-one correspondence with heights of right circular cylinders, i.e. the set of conformal classes is $\mathbb{R}_{>0}$. We will identify conformal classes on $\mathcal{C}$ with points of $\mathbb{R}_{>0}$. We say that the sequence $\{c_n\}$ of conformal classes degenerates if either $c_n \to 0$ or $c_n \to \infty$. The case $c_n \to 0$ corresponds to a pinching geodesic having intersection with two boundary components (i.e. the generatrix of the right circular cylinder). The case $c_n \to \infty$ corresponds to pinching boundary components.

In the case of the Möbius band we also use Theorem 4.7 which implies that in every conformal class on $\mathbb{MB}$ there exists a flat metric with geodesic boundary which is unique if we fix the length of the boundary. Passing to the orientable cover and pulling back the flat metric from $\mathbb{MB}$ we get a flat cylinder with geodesic boundary. Then the discussion in the previous paragraph implies that the conformal classes on $\mathbb{MB}$ are also encoded by $\mathbb{R}_{>0}$. Identifying again conformal classes on $\mathbb{MB}$ with points of $\mathbb{R}_{>0}$ we get two possible cases for a sequence of conformal classes $\{c_n\}$: either $c_n \to 0$ or $c_n \to \infty$. In both cases we say that the sequence $\{c_n\}$ degenerates. The first case corresponds to a pinching geodesic having two points of intersection with boundary. The second case corresponds to the collapsing boundary.

5. Proof of Theorem 1.7.

Negative Euler characteristic. Let Σ be a surface with boundary and $c_n \to c_\infty$ a degenerating sequence of conformal classes. Consider the corresponding sequence of metrics h_n of constant Gauss curvature and geodesic boundary. Then as we have noticed in Subsection 4.4 one can find $s = s_1 + s_2 + s_3$ pinching geodesics of the following three types: s_1 pinching boundary components, s_2 pinching geodesics that have two points of intersection with boundary and s_3 pinching simple closed geodesics that do not intersect the boundary.

We introduce the following notations

- γ_i^n for collapsing geodesics, $i = 1, \dots ,s$. If we do not indicate the superscript then the symbol γ_i stands for the genus;
- $\mathcal{C}_i^n$ for collars of collapsing geodesics, $i = 1, \dots ,s$. Their width are denoted by w_i^n. Moreover, $\mathcal{C}_i^n := \{(t,\theta) \mid 0 \leqslant t < w_i^n,\ 0 \leqslant \theta \leqslant 2\pi\}$ for $1 \leqslant i \leqslant s_1$ and $\mathcal{C}_i^n := \{(t,\theta) \mid -w_i^n < t < w_i^n,\ 0 \leqslant \theta \leqslant 2\pi\}$ for $s_1 + 1 \leqslant i \leqslant s$ (if the geodesic is one-sided then we consider $\mathcal{C}_i^n := \{(t,\theta) \mid -w_i^n < t < w_i^n,\ 0 \leqslant \theta \leqslant 2\pi\}/\sim$, where $\sim$ stands for $(t,\theta) \sim (-t,\pi + \theta)$). Note that geodesics correspond to the line $\{t = 0\}$, the segments $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ are identified for $1 \leqslant i \leqslant s_1$ and for $s_1 + s_2 + 1 \leqslant i \leqslant s$ and they are not identified for $s_1 + 1 \leqslant i \leqslant s_1 + s_2$ and correspond to the segments of intersection with the boundary;
- for $0 < a < w_i^n$, we denote $\mathcal{C}_i^n(0,a)$ the subset $\{(t,\theta) \mid 0 \leqslant t \leqslant a, 0 \leqslant \theta \leqslant 2\pi\} \subset \mathcal{C}_i^n$ for $1 \leqslant i \leqslant s_1$ and for $-w_i^n < a < b < w_i^n$, we denote $\mathcal{C}_i^n(a,b)$ the subset $\{(t,\theta) \mid a \leqslant t \leqslant b, 0 \leqslant \theta \leqslant 2\pi\} \subset \mathcal{C}_i^n$ for $s_1 + 1 \leqslant i \leqslant s$;
- $\Gamma_i^n := \{(\theta,t) \in \mathcal{C}_i^n \mid \theta = 0 \text{ or } \theta = 2\pi\}$ for $s_1 + 1 \leqslant i \leqslant s_1 + s_2$;
- for $-w_i^n < a < b < w_i^n$, we set $\Gamma_i^n(a,b) := \{(\theta,t) \in \Gamma_i^n \mid a \leqslant t \leqslant b\}$ for $s_1 + 1 \leqslant i \leqslant s_1 + s_2$;
- Σ_j^n for the $j-$th connected component of $\Sigma \setminus \cup_{i=1}^s \mathcal{C}_i^n$. We enumerate Σ_j^n by $1 \leqslant j \leqslant M$ such that M denotes the number of Σ_j^n and for all $1 \leqslant j \leqslant m$ one has $\Sigma_j^n \cap \partial\Sigma \neq \varnothing$;
- let $\alpha^n = \cup_{i=1}^{s_1+s_2}\{\alpha_{i,-}^n,\alpha_{i,+}^n\}$, where $0 \leqslant \alpha_i^n < w_i^n$. We denote by $\Sigma_j^n(\alpha^n)$ the connected component of
$$\Sigma \setminus \Big(\bigcup_{i=1}^{s_1+s_2} \mathcal{C}_i^n(\alpha_{i,-}^n,\alpha_{i,+}^n) \cup \bigcup_{i=s_1+s_2+1}^{s} \gamma_i^n \Big)$$
which contains Σ_j^n;
- for $\alpha^n = \cup_{i=1}^{s_1+s_2}\{\alpha_{i,-}^n,\alpha_{i,+}^n\}$, where $0 \leqslant \alpha_i^n < w_i^n$ we set $I_j^n(\alpha^n) = \Sigma_j^n(\alpha^n) \cap \partial\Sigma$ and $I_j^n = \Sigma_j^n \cap \partial\Sigma$ where $1 \leqslant j \leqslant m$;
- we use the notation $a_n \ll b_n$ for two sequences $\{a_n\}$ and $\{b_n\}$ satisfying $a_n, b_n \to +\infty$ and $\frac{a_n}{b_n} \to 0$ as $n \to \infty$.

5.1. Inequality $\geqslant$.

We prove that

$$\liminf_{n\to\infty}\sigma^*_k(\Sigma,c_n)\geqslant \max\Big(\sum_{i=1}^m\sigma^*_{k_i}(\Sigma_{\gamma_i,l_i},c_\infty)+\sum_{i=1}^{s_1+s_2}\sigma^*_{r_i}(\mathbb{D}^2)\Big), \tag{5.1}$$

For this aim we consider the domains $\mathcal{C}^n_i(0,\alpha^n_{i,+})$ for $1\leqslant i\leqslant s_1$, $\mathcal{C}^n_i(\alpha^n_{i,-},\alpha^n_{i,+})$ for $1+s_1\leqslant i\leqslant s_1+s_2$, where $w^n_i-\alpha^n_{i,\pm}\ll w^n_i$, $\alpha^n_{i,\pm}\to\infty$ and the domains $\Sigma^n_j(\alpha^n)$ for $1\leqslant j\leqslant m$. By Lemma 2.8 we have

$$\begin{aligned}\sigma^*_k(\Sigma,c_n)\geqslant \max\Big(&\sum_{i=1}^{s_1}\sigma^{N*}_{r_i}(\mathcal{C}^n_i(0,\alpha^n_{i,+}),\gamma^n_i,c_n)+\\ &+\sum_{i=1+s_1}^{s_1+s_2}\sigma^{N*}_{r_i}(\mathcal{C}^n_i(\alpha^n_{i,-},\alpha^n_{i,+}),\Gamma^n_i(\alpha^n_{i,-},\alpha^n_{i,+}),c_n)+\sum_{j=1}^m\sigma^{N*}_{k_j}(\Sigma^n_j(\alpha^n),I^n_j(\alpha^n),c_n)\Big).\end{aligned} \tag{5.2}$$

For $1\leqslant i\leqslant s_1$ we define the conformal maps $\Psi^n_i\colon(\mathcal{C}^n_i(0,\alpha^n_{i,+}),c_n)\to(\mathbb{D}^2,[g_{can}])$ as

$$\Psi^n_i(t,\theta)=e^{\sqrt{-1}(\theta+\sqrt{-1}t)}.$$

The images of Ψ^n_i are the annuli $\mathbb{D}^2\setminus\mathbb{D}^2_{e^{-\alpha^n_{i,+}}}$ exhausting $\mathbb{D}^2$ as $n\to\infty$. We also note that $\Psi^n_i(\gamma^n_i)=\mathbb{S}^1$.

For $s_1+1\leqslant i\leqslant s_1+s_2$ we define the conformal maps $\Psi^n_i\colon(\mathcal{C}^n_i(\alpha^n_{i,-},\alpha^n_{i,+}),c_n)\to(\mathbb{D}^2,[g_{can}])$ as

$$\Psi^n_i(t,\theta)=\tan\Big(\frac{\theta-\pi+\sqrt{-1}t}{4}\Big).$$

The images of Ψ^n_i that we denote by Ω^n_i exhaust $\mathbb{D}^2$ as $n\to\infty$. We also denote the image of $\Gamma^n_i(\alpha^n_{i,-},\alpha^n_{i,+})$ by $\partial^S\Omega^n_i$. Note that $\partial^S\Omega^n_i$ exhaust $\mathbb{S}^1$ as $n\to\infty$.

Finally, we take restrictions of the diffeomorphisms Ψ^{-1}_n given by Proposition 4.5 to obtain the conformal maps $\check\Psi^n_j\colon(\Sigma^n_j(\alpha^n),c_n)\to(\Sigma_\infty,\Psi^*_nc_n)$ where $1\leqslant j\leqslant m$. Let $\check\Omega^n_j\subset\Sigma_\infty$ be the the image of $\check\Psi^n_j$ and $\partial^S\check\Omega^n_j:=\check\Psi^n_j(I^n_j(\alpha^n))$. The following lemma holds

Lemma 5.1. *Let Σ^∞_j be the connected component $\check\Psi^n_j(\Sigma^n_j)\subset\Sigma_\infty$ where $1\leqslant j\leqslant m$. Then the domains $\check\Omega^n_j$ exhaust Σ^∞_j and $\partial^S\check\Omega^n_j$ exhaust $\partial\Sigma^\infty_j$.*

Proof. Passing to the Schottky double of the surface Σ we immediately deduce this lemma from [**55**, Lemma 5.1]. □

Further, we apply the conformal transformations to (5.2) to get

$$\begin{aligned}\sigma^*_k(\Sigma,c_n)\geqslant\max\Big(&\sum_{i=1}^{s_1}\sigma^{N*}_{r_i}(\mathbb{D}^2\setminus\mathbb{D}^2_{e^{-\alpha^n_{i,+}}},\mathbb{S}^1,[g_{can}])+\\ &+\sum_{i=1+s_1}^{s_1+s_2}\sigma^{N*}_{r_i}(\Omega^n_i,\partial^S\Omega^n_i,[g_{can}])+\sum_{j=1}^m\sigma^{N*}_{k_j}(\check\Omega^n_j,\partial^S\check\Omega^n_j,[(\Psi^n)^*h_n])\Big).\end{aligned} \tag{5.3}$$

It follows from Corollary 2.1 that the first two terms on the right hand side converge to $\sigma_{r_i}(\mathbb{D}^2, [g_{can}])$. To complete the proof we will need the following lemma

Lemma 5.2. *Let $\widehat{\Sigma_j^\infty} \subset \widehat{\Sigma_\infty}$ be a closure of Σ_j^∞, $1 \leqslant j \leqslant m$. Then for all r one has*

$$\liminf_{n\to\infty} \sigma_r^{N*}(\check{\Omega}_j^n, \partial^S\check{\Omega}_j^n, [(\Psi^n)^*h_n]) \geqslant \sigma_r^*(\widehat{\Sigma_j^\infty}, [\widehat{h_\infty}]).$$

We postpone the proof to Section 7.3.

Finally, taking $\liminf_{n\to\infty}$ in (5.3) completes the proof of (5.1).

5.2. Inequality $\leqslant$.

We prove the inverse inequality,

$$\limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n) \leqslant \max\left(\sum_{i=1}^m \sigma_{k_i}^*(\Sigma_{\gamma_i,l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2)\right). \tag{5.4}$$

For this aim we choose a subsequence c_{n_m} such that

$$\lim_{n_m\to\infty} \sigma_k^*(\Sigma, c_{n_m}) = \limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n).$$

Then we relabel the subsequence and denote it by $\{c_n\}$. Therefore, one can choose subsequences without changing the value of lim sup.

Case 1. Suppose that up to a choice of a subsequence the following inequality holds

$$\sigma_k^*(\Sigma, c_n) > \sigma_{k-1}^*(\Sigma, c_n) + 2\pi.$$

Then by [**99**, Theorem 2] in the conformal class c_n there exists a metric g_n of unit boundary length induced from a harmonic immersion with free boundary Φ_n to some $N(n)$-dimensional ball $\mathbb{B}^{N(n)}$, i.e.

$$g_n = \frac{\langle \Phi_n, \partial_{\nu_n}\Phi_n\rangle_{h_n}}{\sigma_k^*(\Sigma, c_n)} h_n$$

and such that $\sigma_k(g_n) = \sigma_k^*(\Sigma, c_n)$. Here the metric h_n is the canonical representative in the conformal class c_n. It is known that for any compact surface the multiplicity of $\sigma_k(g_n)$ is bounded from above by a constant depending only on k and the topology of Σ (see for instance [**31, 54**]). Therefore, one can choose the number $N(n)$ large enough such that $N(n)$ does not depend on n.

Assume that for the sequence $\{c_n\}$ the following inequality holds

$$\limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n) > \max\left(\sum_{i=1}^m \sigma_{k_i}^*(\Sigma_{\gamma_i,l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2)\right). \tag{5.5}$$

For $1 \leqslant i \leqslant s_1$ we consider the conformal map $\Psi_i^n : (\mathcal{C}_i^n, c_n) \to (\mathbb{D}^2, [g_{can}])$ defined as $\Psi_i^n(\theta, t) = e^{\sqrt{-1}(\theta+\sqrt{-1}t)}$. The image of this map is nothing but $\mathbb{D}^2 \setminus \mathbb{D}^2_{e^{-w_i^n}}$ which exhausts $\mathbb{D}^2$ as $n \to \infty$. The image of a pinching geodesic is $\mathbb{S}^1$. Then the map $\Phi_i^n := \Phi_n \circ (\Psi_i^n)^{-1}$:

$\mathbb{D}^2 \setminus \mathbb{D}^2_{e^{-w_i^n}} \to \mathbb{B}^N$ satisfies the *bubble convergence theorem for harmonic maps with free boundary* [**68**, Theorem 1]. Hence, there exist a regular harmonic map with free boundary $\Phi_i : \mathbb{D}^2 \to \mathbb{B}^N$ and some harmonic extensions of non-constant $1/2-$harmonic maps $\omega_1^i, \dots, \omega_{t_i}^i : \mathbb{D}^2 \to \mathbb{B}^N$ such that

$$\int_{\mathbb{D}^2} |\nabla \Phi_i|^2 dv_{g_{can}} + \sum_{j=1}^{t_j} \int_{\mathbb{D}^2} |\nabla \omega_{t_i}^j|^2 dv_{g_{can}} = \lim_{n\to\infty} \int_{\gamma_i^n} ds_{g_n}.$$

We denote $\lim_{n\to\infty} \int_{\gamma_i^n} ds_{g_n}$ by m_i.

Proposition 5.3. *For $s_1+1 \leqslant i \leqslant s_1+s_2$ there exist integers $t_i \geqslant 0$, non-negative sequences $\{a_{i,l}^n\}, \{b_{i,l}^n\}$ with $1 \leqslant l \leqslant t_i$ and a sequence $\{\alpha_i^n\}$ such that*

$$-w_i^n \ll \alpha_{i,-}^n = b_{i,0}^n \ll a_{i,1}^n \ll b_{i,1}^n \ll \ldots \ll a_{i,t_i}^n \ll b_{i,t_i+1}^n \ll a_{i,t_{i+1}}^n = \alpha_{i,+}^n \ll w_i^n$$

and

$$m_{i,l} = \lim_{n\to\infty} L_{g_n}(\Gamma_i^n(a_{i,l}^n, b_{i,l}^n)) > 0.$$

Moreover, there exists a set $J \subset \{1, \dots, m\}$ such that for every $j \in J$ one has

$$m_j = \lim_{n\to\infty} L_{g_n}(I_j^n(\alpha^n)) > 0$$

satisfying

$$\sum_{i=1}^{s_1} m_i + \sum_{i=1}^{s_1+s_2} \sum_{l=s_1+1}^{t_i} m_{i,l} + \sum_{j\in J} m_j = 1,$$

with $s_1 + \sum_{i=s_1+1}^{s_1+s_2} t_i$ is maximal.

Proof. The proof follows the proofs of Claim 16, Claim 17 by [**99**]. Precisely, denying the proposition one can construct $k+1$ test-functions such that $\sigma_k(g_n) \leqslant o(1)$ which contradicts inequality (1.2). The construction of these functions is given in the proofs of Claim 16, Claim 17 by [**99**]. Note that these functions equal 1 on Σ_j^n for every $m+1 \leqslant j \leqslant M$. $\square$

We proceed with considering a sequence $\{d_{i,l}^n\}$ where $s_1+1 \leqslant i \leqslant s_1+s_2$ and $1 \leqslant l \leqslant t_i$ such that

$$\lim_{n\to\infty} L_{g_n}(\Gamma_i^n(a_{i,l}^n, d_{i,l}^n)) = \lim_{n\to\infty} L_{g_n}(\Gamma_i^n(d_{i,l}^n, b_{i,l}^n)) = m_{i,l}/2.$$

Let $q_{i,l}^n \ll a_{i,l}^n$, $q_{i,l}^n \to +\infty$. Consider the conformal maps

$$\Psi_{i,l}^n \colon \left(\mathcal{C}_i^n(a_{i,l}^n - q_{i,l}^n, b_{i,l}^n + q_{i,l}^n), c_n\right) \to (\mathbb{D}^2, [g_{can}])$$

defined as

$$\Psi_{i,l}^n(t,\theta) = \tan\left(\frac{\theta - \pi + \sqrt{-1}(t - t_{i,l}^n)}{4}\right)$$

Let

$$D_{i,j}^n = \Psi_{i,l}^n \left(\mathcal{C}_i^n(a_{i,l}^n - q_{i,l}^n, b_{i,l}^n + q_{i,l}^n)\right)$$

and

$$S^n_{i,j} = \Psi^n_{i,l}\left(\Gamma^n_i(a^n_{i,l} - q^n_{i,l}, b^n_{i,l} + q^n_{i,l})\right)$$

Then $D^n_{i,j}$ exhausts $\mathbb{D}^2$ and $S^n_{i,j}$ exhausts $\mathbb{S}^1$ as $n \to \infty$. We also set

$$\lim_{n\to\infty} L_{(\Psi^n_{i,l})_* g_n}(S^n_{i,j}) = m_{i,l}.$$

Consider the map $\Phi^n_{i,l} = \Phi_n \circ (\Psi^n_{i,l})^{-1} \colon (D^n_{i,j}, S^n_{i,j}) \to (\mathbb{B}^N, \mathbb{S}^{N-1})$. We endow $D^n_{i,j}$ with the metric $(\Psi^n_{i,l})_* g_n$ and $\mathbb{B}^N$ with the Euclidean metric. Then the map $\Phi^n_{i,l}$ is harmonic with free boundary since Φ_n is harmonic with free boundary and $\Psi^n_{i,l}$ is conformal. Moreover, it is shown in [**99**] that the measure $\mathbf{1}_{S^n_{i,j}}\langle \Phi^n_{i,l}, \partial_\nu \Phi^n_{i,l}\rangle_{g_{can}} ds_{g_{can}}$ does not concentrate at the poles $(0,1)$ and $(0,-1)$ of $\mathbb{D}^2$. Indeed, if the measure concentrated at the poles then one would obtain a contradiction with the maximality of $s_1 + \sum_{i=s_1+1}^{s_1+s_2} t_i$.

The exactly same procedure can be carried out for components $\Sigma^n_j(\alpha^n)$, $j \in J$. The only difference is that now we use restrictions of diffeomorphisms Ψ^n given by Proposition 4.5 instead of the explicit harmonic map as above. As a result, one obtains domains $\check{\Omega}^n_j \subset \Sigma_\infty$ and harmonic maps with free boundary $\check{\Phi}^n_j \colon \check{\Omega}^n_j \to \mathbb{B}^N$ such that the measure $\mathbf{1}_{\partial\check{\Omega}^n_j}\langle \Phi^n_{i,l}, \partial_\nu \Phi^n_{i,l}\rangle_{g_{can}} ds_{g_{can}}$ does not concentrate at the marked points of $\widehat{\Sigma_\infty}$.

Now thanks to inequality (5.5) we can construct $k+1$ well-defined test-functions for the Rayleigh quotient of σ_k using the limit functions of the sequences of maps $\hat{\Phi}^n_{i,l}$ and $\hat{\Phi}^n_i$ as it was shown in [**99**]. Precisely, let p_i be the maximal integers such that

$$\frac{\sigma^*_{p_i}(\mathbb{D}^2)}{m_i} < \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n), \tag{5.6}$$

where $1 \leqslant i \leqslant s_1$, $p_{i,l}$ the maximal integers such that

$$\frac{\sigma^*_{p_{i,l}}(\mathbb{D}^2)}{m_{i,l}} < \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n), \tag{5.7}$$

where $s_1 + 1 \leqslant i \leqslant s_1 + s_2$ and p_j the maximal integers such that

$$\frac{\sigma^*_{p_j}(\widehat{\Sigma^\infty_j}, \widehat{c_\infty})}{m_j} < \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),\ j \in J. \tag{5.8}$$

Then one has

$$\sigma^*_{p_i+1}(\mathbb{D}^2) \geqslant m_i \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),\ 1 \leqslant i \leqslant s_1,$$
$$\sigma^*_{p_{i,l}+1}(\mathbb{D}^2) \geqslant m_{i,l} \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),\ s_1 + 1 \leqslant i \leqslant s_1 + s_2$$

and

$$\sigma^*_{p_j+1}(\widehat{\Sigma^\infty_j}, \widehat{c_\infty}) \geqslant m_j \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),\ j \in J.$$

If $\sum_{i=1}^{s_1}(p_i+1) + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i}(p_{i,l}+1) + \sum_{j\in J}(p_j+1) \leqslant k$ then by inequality (5.5) we have

$$\sum_{i=1}^{s_1} \sigma^*_{p_i+1}(\mathbb{D}^2) + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i} \sigma^*_{p_{i,l}+1}(\mathbb{D}^2) + \sum_{j\in J} \sigma^*_{p_j+1}(\widehat{\Sigma^\infty_j}, \widehat{c_\infty}) < \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),$$

which implies $\sum_{i=1}^{s_1} m_i + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i} m_{i,l} + \sum_{j\in J} m_j < 1$ and we arrive at a contradiction with Proposition 5.3. Hence, $\sum_{i=1}^{s_1}(p_i+1) + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i}(p_{i,l}+1) + \sum_{j\in J}(p_j+1) \geqslant k+1$.

Further, let $dv_{g^i_\infty} = \lim_{n\to\infty}(\Psi^n_i)_* dv_{g_n}$, $dv_{g^{i,l}_\infty} = \lim_{n\to\infty}(\Psi^n_{i,l})_* dv_{g_n}$ and $dv_{g^j_\infty} = \lim_{n\to\infty}(\Psi^n_j)^* dv_{g_n}$. Denote by $\widehat{dv_{g^i_\infty}}$, $\widehat{dv_{g^{i,l}_\infty}}$ and $\widehat{dv_{g^j_\infty}}$ the measures induced by the compactification on $\mathbb{D}^2$ for $1 \leqslant i \leqslant s_1$ and $s_1+1 \leqslant i \leqslant s_1+s_2$ and on $\widehat{\Sigma^\infty_j}$ respectively. These measures are well-defined due to the non-concentration argument explained above. Take orthonormal families of eigenfucntions $(\phi^0_i,\ldots,\phi^{p_i}_i)$ in $L^2(\mathbb{D}^2,\widehat{dv_{g^i_\infty}})$ $1 \leqslant i \leqslant s_1$, $(\phi^0_i,\ldots,\phi^{p_{i,l}}_i)$ in $L^2(\mathbb{D}^2,\widehat{dv_{g^{i,l}_\infty}})$ $s_1+1 \leqslant i \leqslant s_1+s_2$ and $(\psi^0_j,\ldots,\psi^{p_j}_j)$ in $L^2(\widehat{\Sigma^\infty_j},\widehat{dv_{g^j_\infty}})$ such that for $0 \leqslant e \leqslant p_i$ the function ϕ^e_i is an eigenfunction with eigenvalue $\sigma_e(\widehat{dv_{g^i_\infty}})$ on $\mathbb{D}^2$, for $0 \leqslant e \leqslant p_{i,l}$ the function ϕ^e_i is an eigenfunction with eigenvalue $\sigma_e(\widehat{dv_{g^{i,l}_\infty}})$ on $\mathbb{D}^2$ and for $0 \leqslant r \leqslant p_j$ the function ψ^r_j is an eigenfunction with eigenvalue $\sigma_r(\widehat{dv_{g^j_\infty}})$ on $\widehat{\Sigma^\infty_j}$. The standard capacity computations (see for instance [**99**, Claim 1]) imply the existence of smooth functions supported in a geodesic ball of a Riemannian manifold and having bounded Dirichlet energy. Let η_i, $\eta_{i,l}$ and η_j be such functions for $(\mathbb{D}^2,\widehat{dv_{g^i_\infty}})$, $(\mathbb{D}^2,\widehat{dv_{g^{i,l}_\infty}})$ and $(\widehat{\Sigma^\infty_j},\widehat{dv_{g^j_\infty}})$ respectively not vanishing everywhere on the boundary. Then we define the desired test-functions as

$$\xi^e_i = (\Psi^n_i)^{-1}\eta_i\phi^e_i,\ 1 \leqslant i \leqslant s_1$$

extended by 0 on Σ,

$$\xi^e_{i,l} = (\Psi^n_{i,l})^{-1}\eta_{i,l}\phi^e_i,\ s_1+1 \leqslant i \leqslant s_1+s_2$$

extended by 0 on Σ and

$$\xi^r_j = \Psi^n_j\eta_j\psi^r_j,\ j\in J$$

extended by 0 on Σ. Note that all these functions have pairwise disjoint supports. Then from the variational characterization of $\sigma_k(g_n)$ one gets

$$\sigma_k(g_n) \leqslant \max\left\{\max_{1\leqslant i\leqslant s_1}\frac{\int_\Sigma|\nabla\xi^e_i|^2_{g_n}dv_{g_n}}{\int_{\partial\Sigma}(\xi^e_i)^2 ds_{g_n}}, \max_{s_1+1\leqslant i\leqslant s_1+s_2}\frac{\int_\Sigma|\nabla\xi^e_{i,l}|^2_{g_n}dv_{g_n}}{\int_{\partial\Sigma}(\xi^e_{i,l})^2 ds_{g_n}}, \max_{j\in J}\frac{\int_\Sigma|\nabla\xi^r_j|^2_{g_n}dv_{g_n}}{\int_{\partial\Sigma}(\xi^r_j)^2 ds_{g_n}}\right\},$$

and passing to lim sup as $n\to\infty$ we get

$$\limsup_{n\to\infty}\sigma^*_k(\Sigma,c_n) \leqslant \max\left\{\max_{1\leqslant i\leqslant s_1}\frac{\sigma^*_{p_i}(\mathbb{D}^2)}{m_i}, \max_{s_1+1\leqslant i\leqslant s_1+s_2}\frac{\sigma^*_{p_{i,l}}(\mathbb{D}^2)}{m_{i,l}}, \max_{j\in J}\frac{\sigma^*_{p_j}(\widehat{\Sigma^\infty_j},\widehat{c_\infty})}{m_j}\right\}$$

which contradicts to (5.7) and (5.8). This means that if inequality (5.5) holds then the sequence $\{c_n\}$ cannot degenerate. We arrived at a contradiction and inequality (5.4) is proved.

Remark 5.4. *Note that if $s_2 = 0$, i.e. there are no pinching geodesics having intersection with boundary components, then we take the set J as $J = \{1,\cdots,m\}$, i.e. we consider $\Sigma^n_j(\alpha^n)$ where $1 \leqslant j \leqslant m$. If all the boundary components are getting pinched then we set $J = \varnothing$ and we only have deal with the functions $\xi^e_i = (\Psi^n_i)^{-1}\eta_i\phi^e_i$ extended by 0 on Σ and $\sigma^*_{p_i}(\mathbb{D}^2)$ where $1 \leqslant i \leqslant s_1$. If $s_1 = s_2 = 0$, i.e. only geodesics of the third type are getting* *pinched then we only have deal with functions* $\xi_j^r = \Psi_j^n \eta_j \psi_j^r$, $j \in J$ *extended by 0 on* Σ *and* $\sigma^*_{p_j}(\widehat{\Sigma_j^\infty}, \widehat{c_\infty})$ *where* $J = \{1, \cdots, m\}$.

Case 2. Assume that up to a choice of a subsequence the following inequality holds

$$\sigma_k^*(\Sigma, c_n) \leqslant \sigma_{k-1}^*(\Sigma, c_n) + 2\pi$$

then we prove inequality (5.4) by induction.

Consider the case $k = 1$ then by inequality 1.2 $\sigma_1^*(\Sigma, c_n) \geqslant 2\pi$. Suppose that up to a choice of a subsequence one has $\sigma_1^*(\Sigma, c_n) > 2\pi$. Then the case $k = 1$ falls under Case 1. Otherwise one has $\limsup_{n\to\infty} \sigma_1^*(\Sigma, c_n) = 2\pi$ and the inequality (5.4) reads as

$$2\pi = \limsup_{n\to\infty} \sigma_1^*(\Sigma, c_n) \leqslant \max\{\sigma_1^*(\Sigma_{\gamma_i, l_i}, c_\infty); 2\pi\},$$

which is true. The base of induction is proved.

Suppose that the inequality holds for all numbers $k' \leqslant k$. We show that it also holds for $k+1$. Indeed, one has

$$\sigma_{k+1}^*(\Sigma, c_n) \leqslant \sigma_k^*(\Sigma, c_n) + 2\pi = \sigma_k^*(\Sigma, c_n) + \sigma_1^*(\mathbb{D}^2)$$

and we get

$$\limsup_{n\to\infty} \sigma_{k+1}^*(\Sigma, c_n) \leqslant \max\left(\sum_{i=1}^m \sigma_{k_i}^*(\Sigma_{\gamma_i, l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2)\right) + \sigma_1^*(\mathbb{D}^2) \leqslant$$

$$\leqslant \max\left(\sum_{i=1}^m \sigma_{k_i}^*(\Sigma_{\gamma_i, l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2)\right),$$

where the maximum is taken over all possible combinations of indices such that

$$\sum_{i=1}^m k_i + \sum_{i=1}^{s_1+s_2} r_i = k+1,$$

since the term $\sigma_1^*(\mathbb{D}^2)$ can be absorbed by one of the terms inside max using inequality (1.1). The proof is complete.

Zero Euler characteristic. The case of the cylinder was essentially considered in [**99**, Section 7.1]. Indeed, it was proved that if the sequence of conformal classes $\{c_n\}$ degenerates then

$$\lim_{n\to\infty} \sigma_k^*(\mathcal{C}, c_n) \leqslant \max_{i_1+\cdots+i_s=k} \sum_{q=1}^s \sigma_{i_q}^*(\mathbb{D}^2) = 2\pi k.$$

Applying then inequality (1.2) one immediately gets that $\lim_{n\to\infty} \sigma_k^*(\mathcal{C}, c_n) = 2\pi k$.

Consider the case of the Möbius band. If the sequence $\{c_n\}$ goes to 0 then it follows from [**99**, Section 7.1] that

$$\lim_{n\to\infty} \sigma_k^*(\mathbb{MB}, c_n) \leqslant \max_{i_1+\cdots+i_s=k} \sum_{q=1}^s \sigma_{i_q}^*(\mathbb{D}^2) = 2\pi k. \tag{5.9}$$

Indeed, we pass to the orientable cover which is a cylinder. Then inequality (5.9) follows from [**99**, Section 7.1, the case $R_\alpha \to 1$ as $\alpha \to +\infty$ in Petrides' notations].

If the sequence $\{c_n\}$ goes to ∞ then we prove that inequality (5.9) also holds. The proof follows the exactly same arguments as in the proof of inequality (5.4). The analog of the case 1 for $\mathbb{MB}$ corresponds to the case of pinching boundary (see Remark (5.4)).

Therefore, in both cases inequality (5.9) holds. Applying inequality (1.2) once again we then get that $\lim_{n\to\infty} \sigma_k^*(\mathbb{MB}, c_n) = 2\pi k$.

6. Proof of Theorem 1.11

For the proof of Theorem 1.11 we will need to choose a "nice" degenerating sequence of conformal classes, i.e. a degenerating sequence of conformal classes such that the limiting space looks as simple as possible.

Lemma 6.1. *Let Σ be a compact surface with boundary of negative Euler characteristic. Then there exists a degenerating sequence of conformal classes such that the limiting space is the disc.*

Proof. The proof is purely topological.

Assume that Σ is orientable. Then we consider collapsing geodesics shown in Figure 6. Passing to the limit when the lengths of all pinching geodesics tend to zero and using the one-point cusps compactification we get an orientable surface of genus 0 with one boundary component, i.e. the disc.

If Σ is non-orientable then we pass to its orientable cover and we consider collapsing geodesics shown in Figure 7 for genus 0 and Figure 8 for genus $\neq 0$ (the pictures are symmetric with respect to the involution changing the orientation, "the antipodal map"). Passing to the limit when the lengths of all pinching geodesics tend to zero and using the one-point cusps compactification we get a disconnected surface with two connected components which are topologically discs. The involution changing the orientation maps one component to another one and hence passing to the quotient by this involution we get just one disc.

$\square$

Now we are ready to prove Theorem 1.11.

Zero Euler characteristic. Let Σ be either the cylinder $\mathcal{C}$ or the Möbius band $\mathbb{MB}$. Then this case immediately follows from Theorem 1.7 by Remark 1.8. Indeed, if $\{c_n\}$ denotes a degenerating sequence of conformal classes on Σ then by Theorem 1.7:

$$I_k^\sigma(\Sigma) \leqslant \lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = 2\pi k.$$

But $I_k^\sigma(\Sigma) \geqslant 2\pi k$ by (1.2). Thus $I_k^\sigma(\Sigma) = \lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = 2\pi k$ and the degenerating sequence $\{c_n\}$ is minimizing.

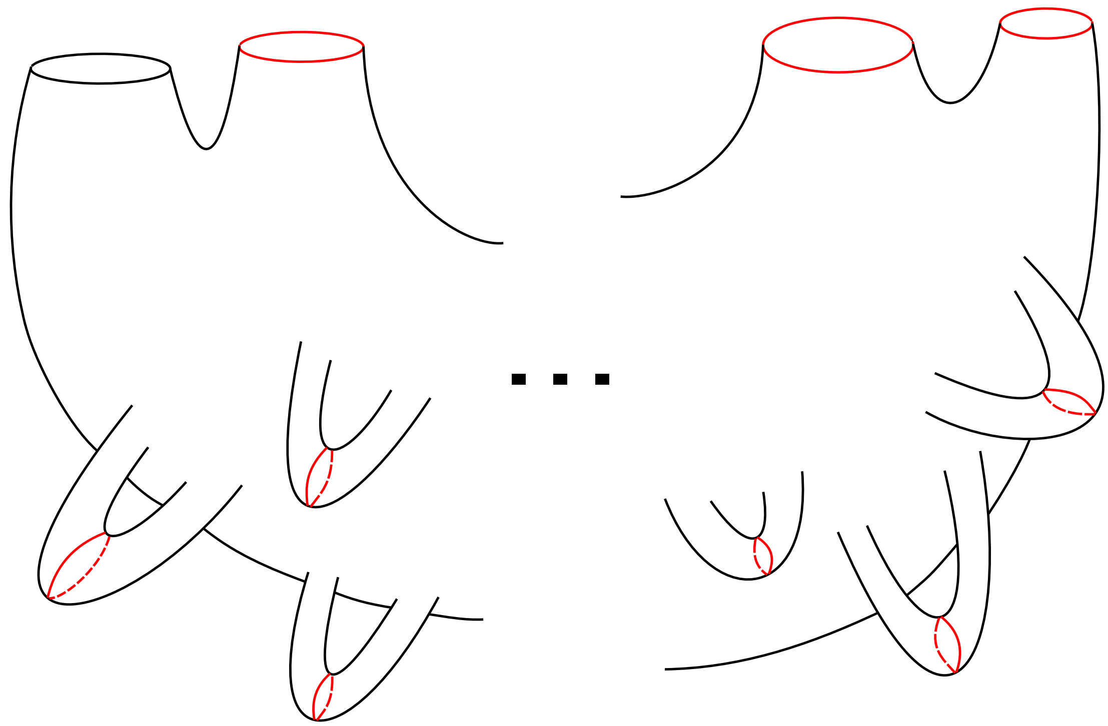

Fig. 6. Orientable surface with boundary. The lengths of all *red* geodesics tend to zero.

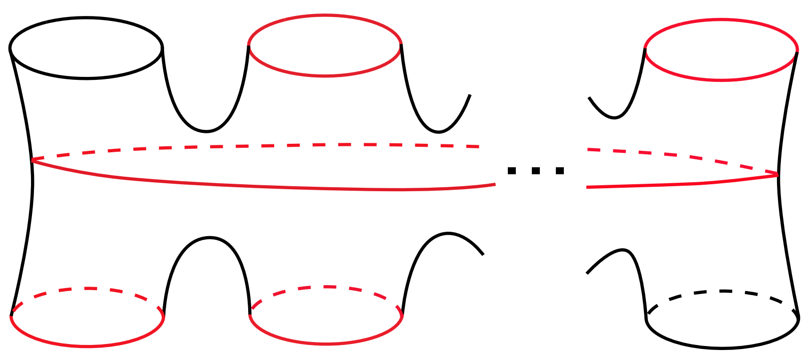

Fig. 7. Orientable cover of a non-orientable surface of genus 0 with boundary. The lengths of all *red* geodesics tend to zero.

Negative Euler characteristic. By Lemma 6.1 there exists a sequence of conformal classes $\{c_n\}$ such that the limiting space $\widehat{\Sigma_\infty}$ is the disc. Then by Theorem 1.7 we have

$$\lim_{n\to\infty}\sigma_k^*(\Sigma,c_n) = \max_{\sum k_j = k}\sum \sigma_{k_j}^*(\mathbb{D}^2).$$

Moreover, we know that $\sigma_k^*(\mathbb{D}^2) = 2\pi k$. Hence,

$$I_k^\sigma(\Sigma) \leqslant \lim_{n\to\infty}\sigma_k^*(\Sigma,c_n) = 2\pi k.$$

Finally, by (1.2) one has $I_k^\sigma(\Sigma) \geqslant 2\pi k$ whence $I_k^\sigma(\Sigma) = 2\pi k$ which completes the proof.

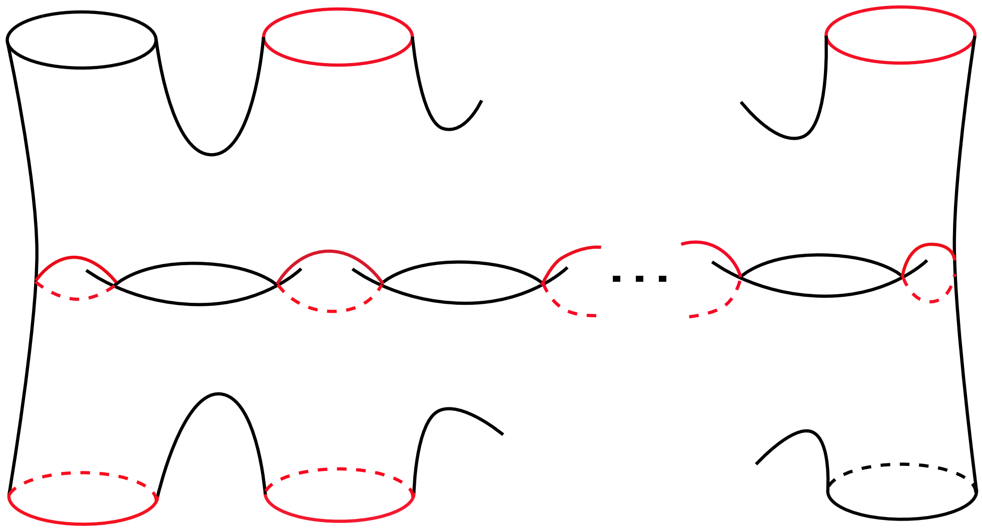

Fig. 8. Orientable cover of a non-orientable surface of genus $\neq 0$ with boundary. The lengths of all *red* geodesics tend to zero.

7. Appendix

7.1. A well-posed problem.

In this section we consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } M, \\ u = g & \text{on } D, \\ \frac{\partial u}{\partial n} = 0 & \text{on } N, \end{cases} \tag{7.1}$$

where (M,h) is a Riemannian manifold with boundary such that $\overline{D} \cup \overline{N} = \partial M$.

Let G be a smooth function such that $G_{|_D} = g$ and consider the function $v = G - u$. Then substituting $u = G - v$ into 7.1 implies:

$$\begin{cases} \Delta v = \Delta G & \text{in } M, \\ v = 0 & \text{on } D, \\ \frac{\partial u}{\partial n} = \frac{\partial G}{\partial n} & \text{on } N. \end{cases} \tag{7.2}$$

We introduce the space $H^1_D(M,h)$ as the closure in H^1-norm of $C^\infty-$functions vanishing on D. For a function $u \in H^1_D(M,h)$ we have the following coercivity inequality:

$$||u||_{L^2(M,h)} \leqslant C||\nabla u||_{L^2(M,h)}, \tag{7.3}$$

with the best constant $C = \frac{1}{\sqrt{\lambda_1^{DN}(M,h)}}$, where $\lambda_1^{DN}(M,h)$ is the first non zero eigenvalue of the mixed problem

$$\begin{cases} \Delta u = \lambda u & \text{in } M, \\ u = 0 & \text{on } D, \\ \frac{\partial u}{\partial n} = 0 & \text{on } N. \end{cases} \tag{7.4}$$

By the Lax-Milgram theorem and by virtue of the inequality (7.3) the problem (7.2) admits a unique solution on the space $H_D^1(M,h)$. Thus, problem 7.1 also has a solution. Moreover, it is easy to see that this solution is unique.

Our aim now is the following lemma.

Lemma 7.1. *Let u satisfy the problem 7.1. Then one has*

$$||u||_{H^1(M,h)} \leqslant C||g||_{H^{1/2}(D,h)}.$$

Proof. The weak formulation of (7.1) reads

$$\int_M \langle \nabla u, \nabla v\rangle dv_h = 0,\ \forall v \in H_D^1(M,h).$$

Let G be any continuation of the function g into M, i.e. $G \in H^1(M,h)$ is any function such that $G_{|_D} = g$. Then substituting $v = u - G$ in the previous identity yields

$$0 = \int_M \langle \nabla u, \nabla u - \nabla G\rangle dv_h = \int_M |\nabla u|^2 dv_h - \int_M \langle \nabla u, \nabla G\rangle dv_h,$$

whence

$$\int_M |\nabla u|^2 dv_h = \int_M \langle \nabla u, \nabla G\rangle dv_h \leqslant \frac{1}{2}\int_M |\nabla u|^2 dv_h + \frac{1}{2}\int_M |\nabla G|^2 dv_h. \tag{7.5}$$

Further, it is easy to see that

$$||u||_{L^2(M,h)} \leqslant ||u - G||_{L^2(M,h)} + ||G||_{L^2(M,h)}.$$

Moreover, since $u - G \in H_D^1(M,h)$ one has

$$||u - G||_{L^2(M,h)} \leqslant C||\nabla u - \nabla G||_{L^2(M,h)} \leqslant C(||\nabla u||_{L^2(M,h)} + ||\nabla G||_{L^2(M,h)}).$$

Substituting it in the previous inequality we get

$$||u||_{L^2(M,h)} \leqslant C(||\nabla u||_{L^2(M,h)} + ||\nabla G||_{L^2(M,h)}) + ||G||_{L^2(M,h)}. \tag{7.6}$$

Plugging (7.5) in (7.6) yields

$$||u||_{L^2(M,h)} \leqslant C||G||_{H^1(M,h)}. \tag{7.7}$$

Finally (7.5) and (7.7) imply

$$||u||_{H^1(M,h)} \leqslant C||G||_{H^1(M,h)} \tag{7.8}$$

for any function $G \in H^1(M,h)$ such that $G_{|_D} = g$.

Lemma 7.2. *The norms*

$$\inf_{G \in H^1(M,h),\ G_{|_D}=g} ||G||_{H^1(M,h)} \text{ and } ||g||_{H^{1/2}(D,h)}$$

are equivalent.

PROOF. By the trace inequality there exists a positive constant C_1 such that for every $G \in H^1(M,h)$ one has

$$||g||_{H^{1/2}(D,h)} \leqslant C_1 ||G||_{H^1(M,h)},$$

which implies:

$$||g||_{H^{1/2}(D,h)} \leqslant C_1 \inf_{G \in H^1(M,h),\ G_{|_D}=g} ||G||_{H^1(M,h)}; \tag{7.9}$$

Further, we construct a continuation $G' \in H^1(M,h)$ of g with the property that there exists a positive constant C_2 such that for every $g \in H^{1/2}(D,h)$ one has:

$$||G'||_{H^1(M,h)} \leqslant C_2 ||g||_{H^{1/2}(D,h)}. \tag{7.10}$$

Let $\tilde{g}$ be any continuation of g on ∂M such that $||\tilde{g}||_{H^{1/2}(N,h)} \leqslant ||g||_{H^{1/2}(D,h)}$. Therefore, $||\tilde{g}||_{H^{1/2}(\partial M,h)} \leqslant \sqrt{2}||g||_{H^{1/2}(D,h)} < \infty$ and $\tilde{g} \in H^{1/2}(\partial M,h)$. Then we take the harmonic continuation of $\tilde{g}$ into M as G'. By [**104**, Proposition 1.7] there exists a positive constant that C_3 such that:

$$||G'||_{H^1(M,h)} \leqslant C_3 ||\tilde{g}||_{H^{1/2}(\partial M,h)}.$$

Since $||\tilde{g}||_{H^{1/2}(\partial M,h)} \leqslant \sqrt{2}||g||_{H^{1/2}(D,h)}$ we get (7.10) with $C_2 = \sqrt{2}C_3$.

Therefore, (7.9) and (7.10) imply:

$$C_2^{-1}||G'||_{H^1(M,h)} \leqslant ||g||_{H^{1/2}(D,h)} \leqslant C_1 \inf_{G \in H^1(M,h),\ G_{|_D}=g} ||G||_{H^1(M,h)},$$

whence

$$C_2^{-1} \inf_{G \in H^1(M,h),\ G_{|_D}=g} ||G||_{H^1(M,h)} \leqslant ||g||_{H^{1/2}(D,h)} \leqslant C_1 \inf_{G \in H^1(M,h),\ G_{|_D}=g} ||G||_{H^1(M,h)},$$

since

$$||G'||_{H^1(M,h)} \geqslant \inf_{G \in H^1(M,h),\ G_{|_D}=g} ||G||_{H^1(M,h)}.$$

And lemma follows. ☐

Finally, taking the infimum over all $G \in H^1(M,h)$ such that $G_{|_D} = g$ in (7.8) and using Lemma 7.2 complete the proof. ☐

7.2. Proofs of propositions of Section 2.

This section contains the proofs of propositions in section 2 analogous to propositions in [**55**, Section 4] whose adaptation to the Steklov setting is almost trivial.

Proof of Lemma 2.4. Let $h^m \in [h]$ be a maximizing sequence of metrics for $\sigma_k^{N*}(\Omega, \partial^S\Omega, [h])$ and $g^m \in [g]$ be a discontinuous metric on Σ defined as $g|_{\Omega_i} = h_i$. By the variational characterization of eigenvalues for all k one has $\sigma_k(\Sigma, g^m) \geqslant \sigma^N(\Omega, h^m)$ since the set of test functions for the Steklov-Neumann eigenvalues $C^0(\Sigma, \{\Omega_i\})$ is larger than the set $C^0(\Sigma)$ of test functions for $\sigma_k(\Sigma, g^m)$. Using the fact that $L_{g^m}(\partial\Sigma) = \sum_i L_{h^m}(\partial^S\Omega_i) \geqslant L_{g^m}(\partial^S\Omega_i)$ for any i and taking the limit as $m \to \infty$ we get

$$\sigma_k^*(\Sigma, \{\Omega_i\}, [g]) \geqslant \sigma_k^{N*}(\Omega, \partial^S\Omega, [h]).$$

Finally by Lemma 2.3 one gets

$$\sigma_k^*(\Sigma, [g]) \geqslant \sigma_k^{N*}(\Omega, \partial^S\Omega, [h]).$$

□

Proof of Proposition 2.6. The proof is similar for both cases. The obvious analog of Lemma 2.5 for the second case holds since its proof follows the exactly same arguments as the proof of Lemma 2.5. For that reason we only provide the proof of Proposition 2.6 for the first case.

Take a maximizing sequence of metrics $\{h_i \mid h_i \in [g|_\Omega]\}$ for the functional $\sigma_k^{N*}(\Omega, \partial^S\Omega, [g])$, i.e.

$$\lim_{i\to\infty} \bar\sigma_k^N(\Omega, \partial^S\Omega, h_i) = \sigma_k^{N*}(\Omega, \partial^S\Omega, [g])$$

Let $h_i = f_i g|_\Omega$, where $f_i \in C_+^\infty(\bar\Omega)$. We then define the metric $\widetilde{h_i} = \widetilde{f}_i g$ on Σ, where $\widetilde{f}_i$ is any positive continuation of the function f_i into Ω^c. It enables us to consider the metric $\rho_\delta\widetilde{h_i}$, where as before

$$\rho_\delta = \begin{cases} 1 & \text{in } \Omega, \\ \delta & \text{in } \Sigma \setminus \Omega. \end{cases}$$

Lemma 2.5 implies

$$\liminf_{\delta\to 0} \sigma_k(\rho_\delta\widetilde{h_i}) \geqslant \sigma_k^N(\Omega, \partial^S\Omega, h_i).$$

Moreover, $L_{\rho_\delta\widetilde{h_i}}(\partial\Sigma) \to L_{h_i}(\partial^S\Omega)$. By Lemma 2.3 we have

$$\sigma_k^*(\Sigma,[g]) = \sigma_k^*(\Sigma,\{\Omega,\Sigma\setminus\Omega\},[g]) \geqslant \liminf_{\delta\to 0}\bar\sigma_k(\rho_\delta\widetilde{h_i}) \geqslant \bar\sigma_k^N(\Omega,\partial^S\Omega,h_i).$$

Therefore, passing to the limit as $i\to\infty$ one gets,

$$\sigma_k^*(\Sigma,[g]) \geqslant \sigma_k^{N*}(\Omega,\partial^S\Omega,[g]).$$

□

Proof of Corollary 2.1. We show that

$$\sigma_k^*(M,[g]) \leqslant \liminf_{n\to\infty}\sigma_k^{N*}(M\setminus K_n,\partial M\setminus\partial K_n,[g]).$$

Let g^m be a maximizing sequence for the functional $\sigma_k^*(M,[g])$. For a fixed m we consider geodesic balls $B_{\epsilon_n}(p_i)$ of radius $\epsilon_n\to 0$ in metric g^m centred at the points $p_1,\ldots,p_l\in M$ such that $K_n\subset\cup_{i=1}^l B_{\epsilon_n}(p_i)$. We see that $M\setminus\cup_{i=1}^l B_{\epsilon_n}(p_i)\subset M\setminus K_n$. Then by Proposition 2.6 one has

$$\begin{aligned}\sigma_k^{N*}(M\setminus K_n,\partial M\setminus\partial K_n,[g]) &\geqslant \sigma_k^{N*}(M\setminus\cup_{i=1}^l B_{\epsilon_n}(p_i),\partial M\setminus\cup_{i=1}^l\partial B_{\epsilon_n}(p_i),[g]) \geqslant\\ &\geqslant \bar\sigma_k^N(M\setminus\cup_{i=1}^l B_{\epsilon_n}(p_i),\partial M\setminus\cup_{i=1}^l\partial B_{\epsilon_n}(p_i),g^m).\end{aligned}\tag{7.11}$$

Note that $L(\partial M\setminus\cup_{i=1}^l\partial B_{\epsilon_n}(p_i),g^m)\to L(\partial M,g^m)$ as $n\to\infty$ and by Lemma 2.1 one has $\sigma_k^N(M\setminus\cup_{i=1}^l B_{\epsilon_n}(p_i),\partial M\setminus\cup_{i=1}^l\partial B_{\epsilon_n}(p_i),g^m)\to\sigma_k(M,g^m)$. Hence, $\bar\sigma_k^N(M\setminus\cup_{i=1}^l B_{\epsilon_n}(p_i),\partial M\setminus\cup_{i=1}^l\partial B_{\epsilon_n}(p_i),g^m)\to\bar\sigma_k(M,g^m)$ as $n\to\infty$. Taking $\liminf_{n\to\infty}$ in (7.11) one then gets

$$\liminf_{n\to\infty}\sigma_k^{N*}(M\setminus K_n,\partial M\setminus\partial K_n,[g]) \geqslant \bar\sigma_k(M,g^m).$$

Passing to the limit as $m\to\infty$ we get the desired inequality.

The inequality

$$\limsup_{n\to\infty}\sigma_k^{N*}(M\setminus K_n,\partial M\setminus\partial K_n,[g]) \leqslant \sigma_k^*(M,[g])$$

follows from Proposition 2.6. This completes the proof. □

Proof of Lemma 2.7. Essentially the idea of the proof comes from the paper [**108**]. We denote by $\partial^S\Omega$ the part of the boundary with the Steklov boundary condition. We also call $\partial^S\Omega$ "Steklov boundary" and $L_g(\partial^S\Omega)$ "the length of Steklov boundary" in metric g.

Inequality $\geqslant$.

Fix the indices $k_i>0$ satisfying $\sum k_i = k$ and consider a maximizing sequence of metrics $\{g_i^m\}$ such that $\bar\sigma_{k_i}^N(\Omega_i,\partial^S\Omega_i,g_i^m)\to\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i])$. One can assume that $\sigma_{k_i}^N(\Omega_i,\partial^S\Omega_i,g_i^m) = \sigma_k^{N*}(\Omega,\partial^S\Omega_i,[g])$. Then, one has

$$L_{g_i^m}(\partial^S\Omega_i)\to\frac{\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i])}{\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])}$$

Let $\{g^m\}$ be a sequence of metrics on Ω defined as $g^m|_{\Omega_i} = g_i^m$. Then for large enough m one has that $\sigma_k^N(\Omega,\partial^S\Omega, g^m) = \sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$, since the spectrum of disjoint union is the union of spectra of each component. By definition of $\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$ we also have

$$\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])L_{g^m}(\partial^S\Omega) = \sigma_k^N(\Omega,\partial^S\Omega,g^m)L_{g^m}(\partial^S\Omega) \leqslant \sigma_k^{N*}(\Omega,\partial^S\Omega,[g]),$$

i.e. $L_{g^m}(\partial^S\Omega) \leqslant 1$. Thus, one has

$$1 \geqslant L_{g^m}(\partial^S\Omega) = \sum_i L_{g_i^m}(\partial^S\Omega_i) \to \frac{\sum_i \sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i])}{\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])}.$$

Passing to the limit $m \to \infty$ yields the inequality.

Inequality $\leqslant$.

Assume the contrary, i.e.

$$\sigma_k^{N*}(\Omega,\partial^S\Omega,[g]) > \max_{\sum_{i=1}^s k_i = k,\ k_i>0} \sum_{i=1}^s \sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i]). \tag{7.12}$$

Consider a maximizing sequence of metrics $\{g^m\}$ of unit total length of Steklov boundary such that $\sigma_k^N(\Omega,\partial^S\Omega,g^m) \to \sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$. Let g_i^m be a restriction of g^m to Ω_i and d_i^m be the largest number satisfying $\sigma_{d_i^m}^N(\Omega_i,\partial^S\Omega_i,g_i^m) < \sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$ and $\limsup_{m\to\infty}\sigma_{d_i^m}^N(\Omega_i,\partial^S\Omega_i,g_i^m) < \sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$. Let L_i^m denote $L_{g_i^m}(\partial^S\Omega_i)$. Then we have $d_i^m \leqslant k$ and $L_i^m \leqslant 1$. Therefore, up to a choice of a subsequence one can assume that $d_i^m = d_i$ does not depend on m and $L_i^m \to L_i$ as $m \to \infty$.

We claim that $\sum_i(d_i+1) \geqslant k+1$. Otherwise, by (7.12) and definition of d_i we have

$$\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])\sum_i L_i \leqslant \sum_i \limsup_{m\to\infty}\bar\sigma_{d_i+1}^N(\Omega_i,\partial^S\Omega_i,g_i^m) \leqslant \sum_i \sigma_{d_i+1}^{N*}(\Omega_i,\partial^S\Omega_i,[g]) <$$
$$< \sigma_k^{N*}(\Omega,\partial^S\Omega,[g]).$$

Moreover, $\sum_i L_i = 1$ since g^m are of unit Steklov boundary length. Thus, we arrive at $\sigma_k^{N*}(\Omega,\partial^S\Omega,[g]) < \sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$, which is a contradiction.

Therefore, the inequality $\sum(d_i+1) \geqslant k+1$ holds. Since the spectrum of a union is a union of spectra, we have $\sigma_k^N(\Omega,\partial^S\Omega,g^m) \in \bigcup_i\{\sigma_0(\Omega_i,g_i^m),\dots,\sigma_{d_i}(\Omega_i,g_i^m)\}$, i.e.

$$\sigma_k^{N*}(\Omega,\partial^S\Omega,g) = \limsup_{m\to\infty}\sigma_k^N(\Omega,\partial^S\Omega,g^m) \leqslant \max_i \limsup_{m\to\infty}\sigma_{d_i}(\Omega_i,g_i^m) < \sigma_k^{N*}(\Omega,\partial^S\Omega,[g]).$$

Since g^m are of unit Steklov boundary length we arrive at a contradiction. □

PROOF OF LEMMA 2.8. Fix indices $k_i \geqslant 0$ such that $\sum_{i=1}^{s'} k_i = k$ and set $I = \{i \,|\, k_i > 0\}$. Let $\Omega_1 = \cup_{i\in I}\overline{\Omega}_i \subset \Sigma$, $\partial^S\Omega_1 = \cup_{i\in I}\partial^S\Omega_i$, $(\Omega_2,h) = \sqcup_{i\in I}(\overline{\Omega}_i,g_{\overline{\Omega}_i})$ and $\partial^S\Omega_2 = \sqcup_{i\in I}\partial^S\Omega_i$. One gets

$$\sigma_k^*(\Sigma,[g]) \geqslant \sigma_k^{N*}(\Omega_1,\partial^S\Omega_1,[g]) \geqslant \sigma_k^{N*}(\Omega_2,\partial^S\Omega_2,[h]) \geqslant \sum_{i\in I}\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g]) =$$

$$= \sum_{i=1}^{s'}\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g]),$$

where we used in order: Proposition 2.6, Lemma 2.4 and Lemma 2.7 and the fact that $\sigma_0^{N*}(\Omega_j,\partial^S\Omega_j,[g]) = 0$ for any j in the last equality. □

7.3. Proof of Lemma 5.2.

Fix $\varepsilon > 0$. An application of Corollary 2.1 to a compact exhaustion of Σ_j^∞ yields the existence of a compact set $K \subset \Sigma_j^\infty \subset \widehat{\Sigma_j^\infty}$ such that

$$|\sigma_r^*(\widehat{\Sigma_j^\infty},[\widehat{h_\infty}]) - \sigma_r^{N*}(K,\partial^S K,[\widehat{h_\infty}])| < \varepsilon,$$

where $\partial^S K = K \cap \partial\Sigma_j^\infty \neq \varnothing$. Since $\check{\Omega}_j^n$ exhaust Σ_j^∞, then for all large enough n one has $K \subset \check{\Omega}_j^n$. Then, by Proposition 2.6

$$\sigma_r^{N*}(\check{\Omega}_j^n,\partial^S\check{\Omega}_j^n,[(\Psi^n)^*h_n]) \geqslant \sigma_r^{N*}(K,\partial^S K,[(\Psi^n)^*h_n]).$$

Taking lim inf of both sides in the above inequality and using Proposition 2.2 yields

$$\liminf_{n\to\infty}\sigma_r^{N*}(\check{\Omega}_j^n,\partial^S\check{\Omega}_j^n,[(\Psi^n)^*h_n]) \geqslant \sigma_r^{N*}(K,\partial^S K,[\widehat{h_\infty}]) > \sigma_r^*(\widehat{\Sigma_j^\infty},[\widehat{h_\infty}]) - \varepsilon.$$

Since ε is arbitrary, this completes the proof.

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